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Math 312 **Final Exam** Jerry L. Kazdan
May 1, 1998 1:30 –3:30

DIRECTIONS: This exam has four parts with a total of 100 points.

Part A:	8 True-False questions (2 points each)	16 points
Part B:	7 Multiple-choice questions (4 points each)	28 points
Part C:	6 Short answer questions (4 points each)	24 points
Part D:	4 Traditional problems (8 points each)	32 points

To receive full credit your solution must be clear and correct. No fuzzy reasoning. Partial credit will *only* be given for the problems in Parts C and D. You have 2 hours. Closed book, no calculators, but you may use one 3×5 card with notes. Please box your answers where appropriate.

NOTE: *To be fair to everyone, those who submit their exam paper late* (after 3:30) *will be "charged" 5 points for every 2 additional minutes.*

For each problem in PART B, record your answer in the table below. Be careful not to fill in more than one answer for any given problem.

B-1
- a) A ◯ B ◯ C ◯
- b) A ◯ B ◯ C ◯
- c) A ◯ B ◯ C ◯
- d) A ◯ B ◯ C ◯
- e) A ◯ B ◯ C ◯

B-2	a ◯	b ◯	c ◯	d ◯	e ◯	f ◯
B-3	a ◯	b ◯	c ◯	d ◯	e ◯	f ◯
B-4	a ◯	b ◯	c ◯	d ◯	e ◯	f ◯
B-5	a ◯	b ◯	c ◯	d ◯	e ◯	f ◯
B-6	a ◯	b ◯	c ◯	d ◯	e ◯	f ◯
B-7	a ◯	b ◯	c ◯	d ◯	e ◯	f ◯

Score	
A	
B	
C-1	
C-2	
C-3	
C-4	
C-5	
C-6	
D-1	
D-2	
D-3	
D-4	
Total	

PART A. *Circle* True (**T**) or False (**F**). 8 problems, 2 points each — but there is a penalty for guessing: Credit = $(2 \times \#correct - \times \#wrong)$, minimum credit $= 0$.

A-1. T F If you have a set of five linearly independent vectors in $\mathbb{R}^5$, then they also span $\mathbb{R}^5$.

A-2. T F The set of all 5×5 symmetric matrices is a linear space.

A-3. T F If C is a square symmetric matrix, then the set of vectors X that satisfy $\langle X, CX \rangle = 0$ is a linear space.

A-4. T F If A is a square matrix with the property that the only solution of $AX = 0$ is $X = 0$, then A is invertible.

A-5. T F If A and B are $n \times n$ matrices with the property that the matrix $M := AB$ is invertible, then both A and B are invertible.

A-6. T F If L is a linear map from $\mathbb{R}^7$ to $\mathbb{R}^4$, then its null space could be *two* dimensional.

A-7. T F If M is a Markov matrix, all of whose elements are *strictly positive*, then for any initial probability vector P_0 the vectors $P_k = M^k P_0$ converge to some limiting vector that does *not* depend on P_0.

A-8. T F If λ is an eigenvalue of the matrix B with corresponding eigenvector V, then $\lambda - 2$ is an eigenvalue of the matrix $B - 2I$ with the *same* eigenvector.

PART B. 7 Multiple-choice questions (4 points each).

B-1. For each of a) — e) below, state *all* of the possibilities that can occur:

A) no solution B) unique solution C) infinitely many solutions

Example: For this example the assertions B) and C) can occur but not A).

A (B) (C) a homogeneous system of 2 equations in 2 unknowns

a) A B C an inhomogeneous system of 7 equations in 7 unknowns

b) A B C an inhomogeneous system of 4 equations in 7 unknowns

c) A B C a homogeneous system of 4 equations in 7 unknowns

d) A B C an inhomogeneous system of 5 equations in 2 unknowns

e) A B C a homogeneous system of 5 equations in 2 unknowns

B-2. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ first rotates the yz-plane by $+90°$ (leaving the x-axis fixed), followed by a reflection across the xy-plane. Find the matrix representation for T in the standard basis for $\mathbb{R}^3$.

a). $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ b). $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ c). $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ d). $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ e). $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$

f). $\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

B-3. If A, B, and C are all $n \times n$ invertible matrices, solve the equation $C^{-1}(I + AX)C = B$ for the matrix X.

a). $X = A^{-1}CBC^{-1} - A^{-1}$ b). $X = A^{-1}B - A^{-1}$ c). $X = A^{-1}CBC^{-1} - I$

d). $X = CBC^{-1}A^{-1} - A^{-1}$ e). $X = BA^{-1} - A^{-1}$ f). $X = A^{-1}C^{-1}BC - B$

B-4. Find *all* values of the constants a, b, and c so that the vectors $(1,1,0,1)$, $(2,-a,0,b)$, and $(0,0,c,0)$ are an *orthogonal* basis for the linear space

$$S = \{(x,y,z,w) \text{ in } \mathbb{R}^4 \text{ with } x+y-2w=0\}$$

a). All a, b with $a-b=2$ and any c
b). All a, b, c with $a-b=2$ and $c \neq 0$
c). $a=6$, $b=4$, and any $c \neq 0$
d). $a=2$, $b=0$, any c
e). $a=2$, $b=0$, any $c \neq 0$
f). These can never be an orthogonal basis for S.

B-5. Consider the vector space P_1 of linear polynomials $ax+b$ with the inner product $\langle \ , \ \rangle$ defined by $\langle p(x),\, q(x)\rangle = \int_0^1 p(x)q(x)\,dx$.
An orthogonal basis of P_1 with this inner product is given by

a). $\{x,\, -2x\}$
b). $\{1,\, 2x-1,\, x+1\}$
c). $\{0,\, 2x-1\}$
d). $\{1,\, x\}$
e). $\{0,\, 1,\, 2x-1\}$
f). $\{x,\, 3x-2\}$

B-6. For which value(s) of the real parameter c, does the polynomial $x^2 + 3y^2 + 2cyz - z^2 + 7$ have a local minimum at the origin?

a). All $c > 0$ b). $|c| < 2$ c). $|c| > 2$ d). $c^2 > -4$ e). $c = 0$ f). No local minimum at the origin for any value of c.

B-7. Let $A := \begin{pmatrix} -3 & b \\ b & -3 \end{pmatrix}$, where b is a real constant. To save time, you are given that the eigenvalues of A are $\lambda = -3 \pm b$. Consider the system of differential equations $\dfrac{dU}{dt} = AU$ for the vector $U(t)$. Find *all* values of the parameter b so that $\lim_{t\to\infty} U(t) = 0$.

a). All $b > 0$ b). $|b| < 3$ b). $b < 9$ d). $b < 3$ e). $b < -3$ f). $|b| \leq 3$

PART C. 6 Short answer questions (4 points each).

C-1. Let $A := \begin{pmatrix} 1 & 4 & 11 & -4 \\ -1 & -2 & -5 & 6 \\ 0 & 4 & 12 & 5 \\ -1 & 2 & 7 & 4 \end{pmatrix}$ and $Y := \begin{pmatrix} -3 \\ 5 \\ 5 \\ 3 \end{pmatrix}$. You are given that the vector $Z := \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$ is a basis for the nullspace of A and that $X_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is a particular solution of $AX = Y$. Find the general solution of $AX = Y$.

C-2. Find a $2\times\ 2$ real matrix A with the property that $A^4 = I$ but $A^2 \neq I$. [HINT: rotation.]

C-3. In $\mathbb{R}^n$, if $X = U + V$, where U and V are perpendicular, show that $\|U\| \le \|X\|$.

C-4. If C is an invertible matrix with eigenvalue λ, show that $1/\lambda$ is an eigenvalue of C^{-1}.

C-5. Let L be a real symmetric matrix. Show that the range of L is orthogonal to the null space of L. [The *range* is sometimes called the *column space*].

C-6. Recall that a 3×3 matrix A is called *anti-symmetric* if $A^t = -A$. On the previous exam you showed that then $\langle X, AX \rangle = 0$ for all vectors X. If λ is a *real* eigenvalue of A, show that $\lambda = 0$.

PART D. 4 Traditional problems (8 points each).

D-1a). Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

b). If $B = \frac{1}{3}A$, find an invertible matrix P and a diagonal matrix D so that $B = PDP^{-1}$.

c). What can you say about $\lim_{k\to\infty} B^k$? (Please briefly justify your assertion.)

D-2. A long queue for tickets at a rock concert at the stadium sees the ticket window open and the first person in line told, "Yes, there are still tickets left for the concert." She tells the person behind her and so on down the line.

However, humans are not completely reliable transmitters of information. If one is told "yes", there is an 90% chance that person will report "yes" to the next in line. On the other hand, being optimistic, if one hears "no", that person will report "yes" 30% of the time.

If the queue is very long, what fraction of them will hear "There are still tickets left"?

D-3. In an experiment you measure the temperature $H(t)$ every six hours on a winter day and get the data:

t	0	6	12	18
H	2	4	8	6

Say you suspect this data should be periodic every 24 hours and have the special form

$$H(t) = a + b\sin\left(\frac{2\pi t}{24}\right) + c\cos\left(\frac{2\pi t}{24}\right).$$

a). Write the (overdetermined) system of equations you would like to solve ideally for a, b, and c.

b). Using the method of least squares write the *normal equations* for the coefficients a, b, and c.

c). Explicitly solve the equations you found in part b).

D-4. Consider the quadratic polynomial $Q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$.
(a) Find a symmetric matrix B such that $Q(X) = \langle X,\, BX \rangle$, where $X = (x_1,\, x_2,\, x_3)$.

(b) It can be shown that the eigenvalues of B are 5, 2 and -1 with corresponding eigenvectors

$$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

Find an orthogonal matrix R such that the change of variable $X = RY$ transforms $Q(X)$ into a quadratic form $Q_{\text{new}}(Y)$ with no cross product terms. Give both R and the new quadratic form $Q_{\text{new}}(Y)$.