

**Homework Set 5, Due Thursday, Feb. 17, 2005***(Late papers will be accepted until 4 PM on Fri. Feb. 18)*

1. a) With the usual Euclidean norm in  $\mathbb{R}^n$  show that for any vectors  $X, Y$ :

$$\|X + Y\|^2 + \|X - Y\|^2 = 2\|X\|^2 + 2\|Y\|^2.$$

Geometrically, this states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the sides.

- b) Show that a parallelogram is a rectangle if and only if its diagonals have the same length:  $\|X + Y\| = \|X - Y\|$ .

2. Strang p. 193 #22, 28

3. Strang p. 202-203, #2, 3, 5, 6, 7

4. Strang p. 204 #13, 16

5. a) Find an orthonormal basis for the plane  $x - y - 2z = 0$  in  $\mathbb{R}^3$ .

- b) In  $\mathbb{R}^3$ , find the  $P$  matrix that projects a vector orthogonally into the plane  $x - y - 2z = 0$  using *both* of the following methods:

METHOD 1. Find an orthonormal basis for the plane and use it to construct the projection.

METHOD 2. Find a vector  $\mathbf{e}$  that is orthogonal to the plane and compute the orthogonal projection  $Q$  along this vector. Then  $P = I - Q$  (why?).

- c) In  $\mathbb{R}^3$  compute the distance from the plane  $x - y - 2z = 3$  to the origin.

- d) In  $\mathbb{R}^3$  compute the distance from the point  $(1, 1, 1)$  to the plane  $x - y - 2z = 3$ .

6. Strang p. 229-230 #15, 16, 18

7. Strang p. 230 #20, 24

8. Strang p. 231 #30, 31,

9. Strang p. 231-231 #33, 34

10. If a square matrix  $A$  has the property that  $A^2 = A$  (this is the general property of a projection) and the null space of  $A$  is zero, show that  $A = I$ .

11. Strang p. 228 #2, 3, 5

12. Let  $\mathbf{n} := (a, b, c) \in \mathbb{R}^3$  be a *unit* vector and  $\mathcal{S}$  the plane of vectors (through the origin) perpendicular to  $\mathbf{n}$ .

a) Show that the *orthogonal projection of  $\mathbf{x}$  in the direction of  $\mathbf{n}$*  can be written in the matrix form

$$\langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (\mathbf{n}\mathbf{n}^T)\mathbf{x} = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where  $\langle \mathbf{x}, \mathbf{n} \rangle$  is the usual inner product,  $\mathbf{n}^T$  is the transpose of the column vector  $\mathbf{n}$ , and  $\mathbf{n}\mathbf{n}^T$  is matrix multiplication.

b) Show that the *orthogonal projection  $P$*  of a vector  $\mathbf{x} \in \mathbb{R}^3$  into  $\mathcal{S}$  is

$$P\mathbf{x} = \mathbf{x} - \langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (I - \mathbf{n}\mathbf{n}^T)\mathbf{x},$$

Apply this to compute the orthogonal projection of the vector  $\mathbf{x} = (1, -2, 3)$  into the plane in  $\mathbb{R}^3$  whose points satisfy  $x - y + 2z = 0$ .

c) Find a formula similar to the previous part for the *orthogonal reflection  $R$*  of a vector across  $\mathcal{S}$ . Then apply it to compute the orthogonal reflection of the vector  $\mathbf{v} = (1, -2, 3)$  across the plane in  $\mathbb{R}^3$  whose points satisfy  $x - y + 2z = 0$ .

d) Find a  $3 \times 3$  matrix that projects a vector in  $\mathbb{R}^3$  into the plane  $x - y + 2z = 0$ .

e) Find a  $3 \times 3$  matrix that reflects a vector in  $\mathbb{R}^3$  orthogonally across the plane  $x - y + 2z = 0$ .

13. a) Let  $v \in \mathbb{R}^n$  be a unit vector and  $Px$  the orthogonal projection of  $x \in \mathbb{R}^n$  in the direction of  $v$ , that is, if  $x = \text{const. } v$ , then  $Px = x$ , while if  $x \perp v$ , then  $Px = 0$ . Show that  $P = vv^T$  (here  $v^T$  is the transpose of the column vector  $v$ ). In matrix notation,  $(P)_{ij} = v_i v_j$ .

b) Continuing, let  $Q$  be the orthogonal projection into the subspace perpendicular to  $v$ . Show that  $Q = I - vv^T$ .

c) Let  $u$  and  $v$  be orthogonal unit vectors and let  $R$  be the orthogonal projection into the subspace perpendicular to both  $u$  and  $v$ . Show that  $R = I - uu^T - vv^T$ .