

DIRECTIONS This exam has two parts, the first has four short computations (5 points each) while the second has seven traditional problems (10 points each).

Part A: Short Computations (4 problems, 5 points each)

A-1. Find a real 2×2 matrix A (other than $A = \pm I$) such that $A^4 = I$.

A-2. Find a function $u(x, y)$ satisfying $\frac{\partial u}{\partial x} - 2u = 0$ with $u(0, y) = \sin(3y)$.

A-3. Say $T(x, y, z) = x^2 + xy + y^3 - z^2$ gives the temperature at the point (x, y, z) . At the point $(1, 1, 1)$, in which direction should one move so that the temperature increases fastest?

A-4. Compute $J := \iint_{\mathbb{R}^2} \frac{1}{[1 + (2x + y + 1)^2 + (x - y + 3)^2]^2} dx dy$.

Part B: Traditional Problems (7 problems, 10 points each)

B-1. If $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$, find an invertible matrix C such that $D := C^{-1}AC$ is a diagonal matrix. Compute A^{50} .

B-2. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a real matrix (not necessarily square). If the nullspace of T is $\{0\}$, show that the matrix T^*T is invertible and positive definite.

B-3. Let a_n be a bounded sequence of real numbers. If $c > 1$, show that the series $\sum \frac{a_n}{n^x}$ converges uniformly in the region $x \geq c$.

B-4. Let $\gamma(t)$ define a smooth curve that does not pass through the origin. If the point $\mathbf{P} = \gamma(t_0)$ is a point on the curve that is closest to the origin (and *not* an end point of the curve), show that the position vector $\gamma(t_0)$ is perpendicular to the tangent vector $\gamma'(t_0)$.

B-5. Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and corresponding orthonormal eigenvectors v_1, \dots, v_n . Show that

$$\lambda_2 = \min_{x \neq 0, x \perp v_1} \frac{\langle x, Ax \rangle}{\|x\|^2}.$$

B-6. Let $f(x)$ be a continuous function for $0 \leq x \leq 1$. Compute $\lim_{n \rightarrow \infty} n \int_0^1 f(x)e^{-3nx} dx$.

B-7. Let \mathcal{D} be a bounded region in the plane, and \mathcal{B} be its boundary (assumed smooth). Let $u(x, y, t)$ be a solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u$ for (x, y) in \mathcal{D} . Say that the temperature $u(x, y, t) = 0$ for all points (x, y) on the boundary \mathcal{B} .

If $E(t) := \frac{1}{2} \iint_{\mathcal{D}} u^2(x, y, t) dx dy$, show that $dE/dt \leq 0$.

More Problems *The following problems were in the first draft of this exam but were deleted since they made the exam way too long.*

M-1. Let $p(x) := \left(\frac{d}{dx}\right)^3 (1-x^2)^3$. Show that p is a polynomial of degree 3 and that it has 3 real distinct zeroes, all lying in the interval $-1 < x < 1$.

M-2. Let $v(x, t) := \int_{x-2t}^{x+2t} g(s) ds$, where g is a continuous function. Compute $\partial v / \partial t$.

M-3. Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a linear map. Show that $\dim \ker(L) - \dim \ker(L^*) = n - k$.

M-4. Let $v(x)$ be a smooth real-valued function for $0 \leq x \leq 1$. If $v(0) = v(1) = 0$ and $v''(x) > 0$ for all $0 \leq x \leq 1$, show that $v(x) \leq 0$ for all $0 \leq x \leq 1$.

M-5. Given any points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ with the x_i 's distinct, show there is a unique cubic polynomial $p(x)$ with the properties that $p(x_i) = y_i$.

M-6. Let $y = f(x, u)$ and $z = g(x, u)$ be smooth functions with, say, $f(x_0, u_0) = y_0$ and $g(x_0, u_0) = z_0$.

- Under what condition(s) can one eliminate x from these equations to express z as $z = H(u, y)$ as a smooth function of u and y near $u = u_0, y = y_0$?
- If $f_x(x_0, u_0) = 1, f_u(x_0, u_0) = -2, g_x(x_0, u_0) = -3,$ and $g_u(x_0, u_0) = 4,$ compute $\partial z / \partial u$ in terms of the derivatives of f and g .

M-7. Let $h(x, y)$ and $f(x)$ be continuous for $0 \leq x \leq 1, 0 \leq y \leq 1$. Let $u_0(x) \equiv 0$ and define $u_k(x), k = 1, 2, \dots,$ recursively by the rule

$$u_{k+1}(x) = f(x) + \int_0^1 h(x, y) u_k(y) dy.$$

If $\max |h(x, y)| \leq c < 1$ show that the $u_k(x)$ converge uniformly for $0 \leq x \leq 1$ to a continuous function $u(x)$ that satisfies

$$u(x) = f(x) + \int_0^c h(x, y) u(y) dy.$$