Some Classical Inequalities

For all of these inequalities there are many methods. We give a sampling.

1. ARITHMETIC-GEOMETRIC MEAN INEQUALITY If $\{b_j\} > 0$, prove the following – and decide when equality holds.

$$(b_1 b_2 \cdots b_n)^{1/n} \le \frac{b_1 + b_2 + \cdots + b_n}{n}. \tag{1}$$

Solution: Here are two approaches. Note that equality holds only if all the b_j's are equal.

METHOD 1. The most naive approach is probably by induction on n. The assertion is clearly true when $n = 1$. Let $B = (b_1 + \cdots + b_n)/n$. Say the desired inequality $b_1 b_2 \cdots b_n \le B^n$ holds for a certain n (our induction hypothesis). Using this we find that

$$(b_1 b_2 \cdots b_{n+1})^{1/(n+1)} \le [B^n b_{n+1}]^{1/(n+1)}, \tag{2}$$

so we will be done if we can show that

$$(B^n b_{n+1})^{1/(n+1)} \le \frac{nB + b_{n+1}}{n+1} \tag{3}$$

(one can interpret this as reducing (1) for $n+1$ terms to the special case when $b_1 = \cdots = b_n = B$). At this point, we could stop since this inequality is a special case of Problem 48 below where $s = B$, $t = b_{n+1}$, and $c = n/(n+1)$. Instead, we proceed directly.

Divide both sides of (3) by B to get the equivalent

$$\left(\frac{b_{n+1}}{B}\right)^{1/(n+1)} \le \frac{n}{n+1} + \frac{1}{n+1}\left(\frac{b_{n+1}}{B}\right). \tag{4}$$

To simplify, let $x := b_{n+1}/B$, so we need to show that $x^{1/(n+1)} \le \frac{n}{n+1} + \frac{1}{n+1}x$ for all $x > 0$ [this is equation (7) with $c = n/(n+1)$]. Since I don't like roots, let $x := y^{n+1}$. After some algebra we must show that

$$ny - (n-1) \le y^n. \tag{5}$$

Using indection on n, the case $n = 0$ is obvious. Thus, assuming (5) we need to show that

$$(n+1)y - n \le y^{n+1} \qquad \text{for all} \qquad y > 0. \tag{6}$$

Using (5) this is equivalent to $y-1 \le y^{n+1}-y^n = (y-1)y^n$, which is obvious if one separately considers the cases $y \ge 1$ and $0 < y < 1$. Equality holds only if $y = 1$, that is, if $B = b_{n+1}$ as claimed.

METHOD 2. See Hardy's *Pure Mathematics*, p. 34.

2. Let $0 < c < 1$. Show that $s^c t^{1-c} < cs + (1-c)t$ for all $s, t > 0$, $s \ne t$ (if $s = t$, then this becomes an equality).

Solution: Dividing both sides by s, this inequality is equivalent to

$$s^{c-1}t^{1-c} < c + (1-c)t/s, \qquad \text{that is} \qquad x^{1-c} < c + (1-c)x, \tag{7}$$

where $0 < x = t/s \ne 1$.

METHOD 1. The function $f(x) := x^{1-c}$ is concave because $f''(x) < 0$. This, the curve lies below its tangent line at $x = 1$. The equation of this tangent line is $y = 1 + (1-c)(x-1) = c + (1-c)x$. Done.

METHOD 2. (very similar) By the mean value theorem applied to $f(x) := x^{1-c}$, we have for some z between 1 and x

$$x^{1-c} - 1 = f(x) - f(1) = f'(z)(x-1) = (1-c)z^{-c}(x-1) < (1-c)x^{-c}(x-1),$$

where in the last inequality one considers the cases $x > 1$ and $x < 1$ separately.

METHOD 3. By elementary calculus, for $a > 0$, $s \ge 0$, the function $\varphi(s) := s^c a^{c-1} - cs$ has its maximum at $s = a$. Thus, $s^c a^{c-1} - cs < (1-c)a$, unless $s = a$.

METHOD 4. Let $s := x^p$, $t := y^q$, $c := 1/p$ and apply Problem #49 below.

3. HÖLDER'S INEQUALITY Let $p, q \ge 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Show that $xy \le \dfrac{x^p}{p} + \dfrac{y^q}{q}$ for all $x, y > 0$.

Solution: METHOD 1. Let $s := x^p$, $t := y^q$, $c := 1/p$ and apply Problem #48 above.

METHOD 2. (similar to #48 method 1). Define u and v by $x := e^{u/p}$ and $y := e^{v/q}$. Since $h(z) := e^z$ is convex, then $h(\lambda u + (1-\lambda)v) \le \lambda h(u) + (1-\lambda)h(v)$ for any $0 \le \lambda \le 1$. If we let $\lambda := 1/p$, then $1/q = 1 - \lambda$ so this gives the desired inequality.

METHOD 3. By elementary calculus, for $a > 0$ and $x \ge 0$ the maximum of $g(x) := ax - x^p/p$ occurs at $x = a^{1/(p-1)}$. Thus $ax \le x^p/p + a^q/q$.

METHOD 4. We'll show that on the set $uv = 1$ one has $f(u,v) := \frac{u^p}{p} + \frac{v^q}{q} \geq 1$. Since on the constraint $uv = 1$ the function $f(u,v)$ blows up as u or v tend to infinity, we know theree is a global min at a finite point.

To find it we use Lagrange multipliers and consider $F(u,v) := f(u,v) + \lambda(uv-1)$. Then the conditions $0 = F_u = u^{p-1} + \lambda v$ and $0 = F_v = v^{q-1} + \lambda u$ along with the constraint $uv = 1$ imply (after a calculation) that $u = v = 1$. Since there is only one critical point, this must be the global minimum: $f(u,v) \geq f(1,1) = 1$.

The substitutions $u^p = \frac{x^p}{xy}$, $v^q = \frac{y^q}{xy}$, that is, $u = \frac{x^{1/q}}{y^{1/p}}$, $v = \frac{y^{1/p}}{x^{1/q}}$ then give the desired inequality.