

# Calculus Problems

Math 504 – 505

Jerry L. Kazdan

1. Sketch the points  $(x, y)$  in the plane  $\mathbb{R}^2$  that satisfy  $|y - x| \leq 2$ .
2. A certain function  $f(x)$  has the property that  $\int_0^x f(t) dt = e^x \cos x + C$ . Find both  $f$  and the constant  $C$ .
3. Compute  $\lim_{x \rightarrow 0} \left( \frac{\cos x}{\cos 2x} \right)^{1/x^2}$ .
4. Sketch the curve that is defined implicitly by  $x^3 + y^3 - 3xy = 0$ . Calculate  $y'(0)$ .
5. Calculate  $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$ .
6. Determine the indefinite integral  $\int \log(1 + x^2) dx$ .
7. Let  $f(x)$  be a smooth function for  $0 \leq x \leq 1$ . If  $f'(x) = 0$  for all  $0 \leq x \leq 1$ , what can you conclude? Prove all your assertions.
8. Solve the initial value problem  $(1 + e^x)yy' = e^x$  with  $y(1) = 1$ .
9. Let the continuous function  $f(\theta)$ ,  $0 \leq \theta \leq 2\pi$  represent the temperature along the equator at a certain moment, say measured from the longitude at Greenwich.. Show there are antipodal points with the *same* temperature.
10. a) Let  $g(x) := x^3(1 - x)$ . Without computation, show that  $g'''(c) = 0$  for some  $0 < c < 1$ .  
b) Let  $h(x) := x^3(1 - x)^3$ . Show that  $h'''(x)$  has exactly three distinct roots in the interval  $0 < x < 1$ .  
c) Let  $p(x) := \left( \frac{d}{dx} \right)^4 (1 - x^2)^4$ . Show that  $p$  is a polynomial of degree 4 and that it has 4 real distinct zeroes, all lying in the interval  $-1 < x < 1$ .

11. In  $\mathbb{R}^2$ , let  $Q_1 = (x_1, y_1)$ ,  $Q_2 = (x_2, y_2)$ , and  $Q_3 = (x_3, y_3)$ , where  $x_1 < x_2 < x_3$ .
- Show there is a unique quadratic polynomial  $p(x)$  that passes through these points:  $p(x_j) = y_j$ ,  $j = 1, 2, 3$ .
  - If  $y_1 > y_2$  and  $y_3 > y_2$  and  $f(x)$  is any smooth function that passes through these three points, show there is some point  $c \in (x_1, x_3)$  where  $f''(c) > 0$ . Even better, for some  $c$ ,  $f''(c) \geq p''$ , so  $p''$  is the optimal constant. [Remark: It is enough to consider the special case where  $x_2 = 0$  and  $y_2 = 0$ . Then write  $x_1 = -a < 0$ ,  $x_3 = b > 0$ ].
12. a) If  $f(x) > 0$  is continuous for all  $x \geq 0$  and the improper integral  $\int_0^\infty f(x) dx$  exists, then  $\lim_{x \rightarrow \infty} f(x) = 0$ . Proof or counterexample.
- b) If  $f(x) > 0$  is continuous for all  $x \geq 0$  and the improper integral  $\int_0^\infty f(x) dx$  exists, then  $f(x)$  is bounded. Proof or counterexample.
13. Find *explicit* rational functions  $f(x)$  and  $g(x)$  with the following Taylor series:  $f(x) = \sum_1^\infty nx^n$ ,  $g(x) = \sum_1^\infty n^2x^n$ .
14. a) Let  $x = (x_1, x_2)$  be a point in  $\mathbb{R}^2$  and consider  $\int_{\mathbb{R}^2} \frac{1}{(1 + \|x\|^2)^p} dx$ . For which  $p$  does this improper integral converge?
- This integral can be computed explicitly. Do so.
  - Repeat part a) where  $x \in \mathbb{R}^3$  and the integral is over  $\mathbb{R}^3$  instead of  $\mathbb{R}^2$ .
15. Compute  $\iint_{\mathbb{R}^2} \frac{1}{[1 + (2x + y + 1)^2 + (x - y + 3)^2]^2} dx dy$ .
16. Let  $v(x, t) := \int_{x-2t}^{x+2t} g(s) ds$ , where  $g$  is a continuous function. Compute  $\partial v / \partial t$  and  $\partial v / \partial x$ .
17. Let  $H(t) := \int_{a(t)}^{b(t)} f(x, t) dx$ , where  $a(t)$ ,  $b(t)$ , and  $f(x, t)$  are smooth functions of their variables. Compute  $dH/dt$ .
18. a) Let  $p(x) := x^3 + cx + d$ , where  $c$ , and  $d$  are real. Under what conditions on  $c$  and  $d$  does this has three distinct real roots? [ANSWER:  $c < 0$  and  $d^2 < -4c^3/27$ ].

- b) Generalize to the real polynomial  $p(x) := ax^3 + bx^2 + cx + d$  ( $a \neq 0$ ) by a change of variable reducing to the above special case.
19. If  $b \geq 0$ , show that for every real  $c$  the equation  $x^5 + bx + c = 0$  has exactly one real root. What if  $b < 0$ ? Say as much as you can.
20. Let  $f(t)$  and  $g(t)$  be smooth increasing functions of  $t \in \mathbb{R}$ . Proof or counterexample:
- $f(t) + g(t)$  is an increasing functions of  $t$ .
  - $f(t)g(t)$  is an increasing functions of  $t$ .
  - If  $f(t) > 0$  and  $g(t) > 1$  then  $f(t)^{g(t)}$  is an increasing functions of  $t$ .
21. Let a smooth function  $g(x)$  have the three properties:  $g(0) = 2$   $g(1) = 0$   $g(4) = 6$ . Show that at some point  $0 < c < 4$  one has  $g''(c) > 0$ . Better yet, find a number  $m > 0$  so that  $g''(c) \geq m > 0$ .  
Is it true that  $g''$  must be positive at at least one point  $0 < c < 1$ ? Proof or counterexample.
22. Let  $f(x)$  be a differentiable function for all real  $x$  with the property that  $f'(x) < 1$  for all  $x$ . Show has at most one *fixed point*, that is, at most one point  $c$  where  $f(c) = c$ .
23. Let  $g$  be a differentiable function with the properties  $g(a) = 0$ ,  $g(b) = 0$ , and  $g'(x) \geq 0$  for all  $x \in [a, b]$ . What can you deduce about  $g(x)$  for  $x \in [a, b]$ ? Justify your conclusions.
24. Let  $v(x)$  be a smooth real-valued function for  $0 \leq x \leq 1$ . If  $v(0) = v(1) = 0$  and  $v''(x) > 0$  for all  $0 \leq x \leq 1$ , show that  $v(x) \leq 0$  for all  $0 \leq x \leq 1$ .
25. If a smooth curve  $y = f(x)$  has the property that  $f''(x) > 0$ , show that the chord joining two points of the curve lies above the curve:

$$tf(b) + (1-t)f(a) \geq f(tb + (1-t)a) \quad \text{for all } 0 \leq t \leq 1.$$

26. a) Find an integer  $N$  so that  $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{N}} > 100$ .
- b) Find an integer  $N$  so that  $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} > 100$ .

27. Let  $c$  be any complex number. Show that  $\lim_{n \rightarrow \infty} \frac{c^n}{n!} = 0$ .
28. a) Show that  $\sin x$  is not a polynomial.  
 b) Show that  $\sin x$  is not a rational function, that is, it cannot be the quotient of two polynomials.  
 c) Let  $f(t)$  be periodic with period 1, so  $f(t+1) = f(t)$  for all real  $t$ . If  $f$  is not a constant, show that it cannot be a rational function. that is,  $f$  cannot be the quotient of two polynomials.  
 d) Show that  $e^x$  is not a rational function.
29. Let  $f(x)$  be a differentiable function of  $x := (x_1, x_2, x_3)$  for all  $x \in \mathbb{R}^3$ . If  $f$  is *homogeneous of degree  $k$*  in the sense that  $f(cx) = c^k f(x)$  for all  $c > 0$ , show that  $x \cdot \nabla f(x) = kf(x)$  (Euler).
30. The *Gamma function* is defined by  $\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt$ .  
 a) For which real  $x$  does this improper integral converge?  
 b) Show that  $\Gamma(x+1) = x\Gamma(x)$  and deduce that  $\Gamma(n+1) = n!$  for any integer  $n \geq 0$ .
31. Say  $\gamma(t) : \mathbb{R} \rightarrow \mathbb{R}^2$  defines a smooth curve in the plane.  
 a) If  $\gamma(0) = 0$  and  $\|\gamma'(t)\| \leq c$ , show that for any  $T \geq 0$ ,  $\|\gamma(T)\| \leq cT$ . Moreover, show that equality can occur if and only if one has  $\gamma(t) = cvt$  where  $v$  is a unit vector that does not depend on  $t$ .  
 b) If  $\gamma(0) = 0$ ,  $\gamma'(0) = 0$  and  $\|\gamma''(t)\| \leq 12$ , give an upper bound estimate for  $\|\gamma(2)\|$ . When can this upper bound be achieved?
32. Let  $\mathbf{r}(t)$  define a smooth curve that does not pass through the origin.  
 a) If the point  $\mathbf{a} = \mathbf{r}(t_0)$  is a point on the curve that is closest to the origin (and *not* an end point of the curve), show that the position vector  $\mathbf{r}(t_0)$  is perpendicular to the tangent vector  $\mathbf{r}'(t_0)$ .  
 b) What can you say about a point  $\mathbf{b} = \mathbf{r}(t_1)$  that is *furthest* from the origin?
33. Consider two smooth plane curves  $\gamma_1, \gamma_2 : (0, 1) \rightarrow \mathbb{R}^2$  that do not intersect. Suppose  $P_1$  and  $P_2$  are points on  $\gamma_1$  and  $\gamma_2$ , respectively, such that the distance  $|P_1P_2|$  is minimal. Prove that the straight line  $P_1P_2$  is normal to *both* curves.

34. Let  $h(x, y, z) = 0$  define a smooth surface in  $\mathbb{R}^3$  and let  $P := (a, b, c)$  be a point *not* on the surface. If  $Q : (x, y, z)$  is a point on the surface that is closest to  $P$ , show that the line  $PQ$  is perpendicular to the tangent plane to the surface at  $Q$ .
35. Let  $\mathbf{r}(t)$  describe a smooth curve and let  $\mathbf{V}$  be a fixed vector. If  $\mathbf{r}'(t)$  is perpendicular to  $\mathbf{V}$  for all  $t$  and if  $\mathbf{r}(0)$  is perpendicular to  $\mathbf{V}$ , show that  $\mathbf{r}(t)$  is perpendicular to  $\mathbf{V}$  for all  $t$ .
36. Let  $f(s)$  be any differentiable function of the real variable  $s$ . Show that  $u(x, t) := f(x + 3t)$  has the property that  $u_t = 3u_x$ . Show that  $u$  also satisfies the wave equation  $u_{tt} = 9u_{xx}$ .
37. Let  $u(x, y)$  be a smooth function.
- If  $u_x = 0$  with  $u(0, y) = \sin(3y)$ , find  $u(x, y)$ .
  - If  $u_x = 2xy$  with  $u(0, y) = \sin(3y)$ , find  $u(x, y)$ .
  - If  $u_x + u_y = 0$  with  $u(0, y) = \sin(3y)$ , find  $u(x, y)$ . Is there more than one such function?
  - If  $u_x + u_y = 3 - 2xy$  with  $u(0, y) = \sin(3y)$ , find  $u(x, y)$ . Is there more than one such function?
  - If  $u_x - 2u_y = 0$  with  $u(0, y) = \sin(3y)$ , find  $u(x, y)$ . Is there more than one such function?
38. Let  $\mathbf{r} := x\mathbf{i} + y\mathbf{j}$  and  $\mathbf{V}(x, y) := p(x, y)\mathbf{i} + q(x, y)\mathbf{j}$  be (smooth) vector fields and  $C$  a smooth curve in the plane. In this problem  $I$  is the line integral  $I = \int_C \mathbf{V} \cdot d\mathbf{r}$ . For each of the following, either give a proof or give a counterexample.
- If  $C$  is a vertical line segment and  $q(x, y) = 0$ , then  $I = 0$ .
  - If  $C$  is a circle and  $q(x, y) = 0$ , then  $I = 0$ .
  - If  $C$  is a circle centered at the origin and  $p(x, y) = -q(x, y)$ , then  $I = 0$ .
  - If  $p(x, y) > 0$  and  $q(x, y) > 0$ , then  $I > 0$ .
39. Let  $C$  denote the unit circle centered at the origin of the plane, and  $D$  denote the circle of radius 5 centered at  $(2, 1)$ , both oriented counterclockwise. Let  $Q$  denote the ring region between these curves. If a vector field  $\mathbf{V}$  satisfies  $\text{div } \mathbf{V} = 0$ , show that the line integral  $\int_C \mathbf{V} \cdot \mathbf{N} ds = \int_D \mathbf{V} \cdot \mathbf{N} ds$ . [This extends immediately to the situation where  $C$  and  $D$  are more general curves and  $Q$  is the region between them. For fluid flow it is an expression of *conservation of mass*, since  $\text{div } \mathbf{V} = 0$  means there are no sources or sinks in the region  $Q$ .]

40. (Integration by Parts for Multiple Integrals) Let  $\mathbf{F}$  be a smooth vector field and  $u$  a smooth scalar-valued function.

- Prove the identity  $\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u\nabla \cdot \mathbf{F}$ . Compare this with the special case of a function of one variable.
- Let  $\mathcal{D}$  be a bounded region in the plane whose boundary is the curve  $C$  with unit outer normal  $\mathbf{N}$ . Also, let  $u$  be a scalar-valued function, and  $\mathbf{F}$  a vector field. Prove the identity

$$\iint_{\mathcal{D}} u\nabla \cdot \mathbf{F} dA = \oint_C u\mathbf{F} \cdot \mathbf{N} ds - \iint_{\mathcal{D}} \nabla u \cdot \mathbf{F} dA.$$

Notice that for a function of one variable with  $\mathcal{D}$  being the interval  $\{a < x < b\}$ , this reduces precisely to the usual formula for integration by parts.

- Generalize this formula to the case where  $\mathcal{D}$  is a bounded (solid) region in three dimensional space.
- One frequently uses this with  $\mathbf{F} = \nabla v$ . Show the above formula for integration by parts becomes (say in two dimensions)

$$\iint_{\mathcal{D}} u\nabla \cdot \nabla v dA = \oint_C u\nabla v \cdot \mathbf{N} ds - \iint_{\mathcal{D}} \nabla u \cdot \nabla v dA.$$

This is *Green's theorem*. To what does this reduce for functions on one variable?

- As a short application using this, say  $u(x,y)$  is a *harmonic function* in a bounded region  $\mathcal{D}$ , so  $\Delta u := \nabla \cdot \nabla u = u_{xx} + u_{yy} = 0$ . One can think of  $u(x,y)$  as being the equilibrium temperature of  $\mathcal{D}$ . Let  $C$  is the boundary of  $\mathcal{D}$ . If  $u = 0$  on  $C$ , it is plausible that one must have  $u(x,y) = 0$  throughout  $\mathcal{D}$ . Show how this follows from the above formula. What is the analogous assertion for functions of one variable, where a harmonic function is just a solution of  $u'' = 0$ ?

41. Let  $\mathcal{D}$  be a bounded region in the plane, and let  $\mathcal{B}$  be its boundary.

- Use the divergence theorem (or any related formula you know) to show that for any smooth function  $v(x,y)$

$$\iint_{\mathcal{D}} \Delta v dx dy = \int_{\mathcal{B}} \frac{\partial v}{\partial N} ds$$

where  $\partial v / \partial N := \nabla v \cdot \mathbf{n}$  is the outer normal directional derivative on  $\mathcal{B}$ .

- Let  $u(x,y,t)$  be a solution of the heat equation  $u_t = \Delta u$  for  $(x,y)$  in  $\mathcal{D}$ . Assume that the boundary,  $\mathcal{B}$ , is *insulated*, so the outer normal derivative there is zero:  $\frac{\partial u}{\partial N} = 0$  on  $\mathcal{B}$ .

Show that  $Q(t) := \iint_{\mathcal{D}} u(x,y,t) dx dy$  is a constant.

42. Continuing the notation of the previous problem, say that instead the temperature  $u(x, y, t) = 0$  for all points  $(x, y)$  on the boundary  $\mathcal{B}$ .
- Show that the function  $E(t) := \frac{1}{2} \int_{\mathcal{D}} u^2(x, y, t) dx dy$  has the property that  $dE/dt \leq 0$ .
  - Use this to show that with these zero boundary conditions, if the initial temperature is zero,  $u(x, y, 0) = 0$ , then  $u(x, y, t) = 0$  for all  $t \geq 0$ .
43. Let  $u(x, y, t)$  describe the motion of a vibrating drumhead  $\mathcal{D}$ . A reasonable mathematical model shows that  $u$  satisfies the *wave equation*  $u_{tt} = \Delta u$  in  $\mathcal{D}$  with *boundary condition*  $u(x, y, t) = 0$  for all  $(x, y)$  on the boundary  $\partial\mathcal{D}$ .

Physical reasoning leads one to define the *energy* as

$$E(t) := \frac{1}{2} \iint_{\mathcal{D}} (u_t^2 + |\nabla u|^2) dA.$$

- Show that energy is conserved:  $E(t) = E(0)$ . [HINT: Show  $dE/dt = 0$ .]
  - If in addition one knows that the initial position  $u(x, y, 0) = 0$  and that the *initial velocity*  $u_t(x, y, 0) = 0$ , show that  $E(t) = 0$  for all  $t$  and deduce that  $u(x, y, t) \equiv 0$ . [This is hardly a surprise on physical grounds, but it should be interpreted as reassuring us that this mathematical model is indeed reasonably correct.]
44. If  $h'(t) \leq ch(t)$ , where  $c$  is a constant, show that  $h(t) \leq e^{ct}h(0)$  for all  $t \geq 0$ .
45. Say  $u(t)$  satisfies  $u'' + b(t)u' + c(t)u = 0$ , where  $b(t)$  and  $c(t)$  are bounded functions. Let  $E(t) := \frac{1}{2}(u'^2 + u^2)$ .
- Show that  $E'(t) \leq \gamma E(t)$ , where  $\gamma$  is a constant.
  - Deduce that if  $u(0) = 0$  and  $u'(0) = 0$ , then  $u(t) = 0$  for all  $t$ .
46. Let  $\mathcal{V} := \{u(x) \in C^2(\mathbb{R}) \mid u'' + u = 0\}$ . Prove that  $\dim \mathcal{V} = 2$ . Prove all of your assertions in detail.

47. The solutions to the following matrix differential equation

$$X' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} X$$

form a vector space. Find a basis for this vector space.

48. Consider the differential equation  $X'(t) = AX(t)$  where

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Which of the following assertions are correct—and why?

- There is a solution of the form  $X(t) = U$ , where  $U$  is a real constant (non-zero) vector.
- There is a solution of the form  $X(t) = Ve^{2t}$ , where  $V$  is a real constant (non-zero) vector.
- There is a solution of the form  $X(t) = Ve^{-2t}$ , where  $V$  is a real constant (non-zero) vector.
- There is a complex solution of the form  $X(t) = We^{it}$ , where  $W$  is a constant (non-zero) vector.
- All of the solutions of this equation remain bounded as  $t \rightarrow \infty$ .

49. Consider the *second order* differential equation  $X'' = AX$  where  $A$  is a symmetric  $2 \times 2$  matrix.

- Find the general solution if  $A = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$ .
- Find the general solution if  $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$ . [Suggestion: First diagonalize  $A$ , so  $D := R^{-1}AR$  is diagonal. Then make the change of variables  $X = RY$  to obtain a simpler differential equation for  $Y(t)$ .]
- Find the general solution if  $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ .

50. For which complex numbers  $z$  does the series  $\sum_1^\infty ne^{-nz}$  converge?

51. a) Let  $u(x_1, \dots, x_n)$  be a smooth function that depends only on the distance  $r = \sqrt{x_1^2 + \dots + x_n^2}$ . Show that

$$\frac{\partial^2 u}{\partial x_j^2} = \frac{x_j^2}{r^2} \frac{d^2 u}{dr^2} + \frac{(r^2 - x_j^2)}{r^3} \frac{du}{dr}, \quad \text{and hence} \quad \Delta u = u_{rr} + \frac{n-1}{r} u_r.$$

- b) Find all harmonic functions (these are the solutions of  $\Delta u = 0$ ) that depend only on  $r$ .
52. a) Find the equation of the tangent plane to the surface  $x^2 + xy + y^3 - z^2 = 2$  at the point  $(1, 1, 1)$ .
- b) Say the function  $T = x^2 + xy + y^3 - z^2$  gives the temperature at the point  $(x, y, z)$ . At the point  $(1, 1, 1)$ , in which direction should one move so that the temperature increases fastest?
53. Let  $\psi(t)$  be a scalar-valued function with a continuous derivative for  $0 < t < \infty$  and let  $\mathbf{X} = (x, y, z) \in \mathbb{R}^3$ . Define the vector field  $\mathbf{F}(\mathbf{X}) := \psi(\|\mathbf{X}\|)\mathbf{X}$  for all  $\mathbf{X} \neq \mathbf{0}$ . Show that this vector field is conservative by finding a scalar-valued function  $\phi(r)$  with the property that  $\mathbf{F}(\mathbf{X}) := \nabla\phi(\|\mathbf{X}\|)$ . In particular, this shows that *every central force field is conservative*.

54. Let  $\mathcal{D}$  be a bounded region in the plane with smooth boundary  $\mathcal{B}$ . Show that

$$\text{Area}(\mathcal{D}) = \frac{1}{2} \int_{\mathcal{B}} x dy - y dx.$$

Use this to find the area inside the ellipse  $(x, y) = (a \cos \theta, b \sin \theta)$  for  $0 \leq \theta \leq 2\pi$ .

55. If  $\{b_j\} > 0$ , prove the arithmetic-geometric mean inequality

$$(b_1 b_2 \cdots b_n)^{1/n} \leq \frac{b_1 + b_2 + \cdots + b_n}{n}.$$

When is there equality?

56. Let  $0 < c < 1$ . Show that  $s^c t^{1-c} < cs + (1-c)t$  for all  $s, t > 0$ ,  $s \neq t$  (if  $s = t$ , then this becomes an equality).

57. Let  $p, q \geq 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Show that  $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$  for all  $x, y > 0$ .

58. Let  $P_1, \dots, P_n$  be  $n \geq 3$  points on a circle and let  $Q$  be the polygon obtained by connecting these successive points. How should the points be situated to maximize the area of  $Q$ ?

59. a) Find a smooth function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  that has exactly three critical points, all non-degenerate, one being a local max, one a local min, and the third a saddle.  
 b) Show there is no such  $f(x, y)$  of the form  $f(x, y) = g(x) + h(y)$ .

60. Compute  $\lim_{\lambda \rightarrow \infty} \int_0^1 |\sin \lambda x| dx$  (part of the problem is to show that the limit exists).

61. a) State what it means for a real-valued function defined on the closed, bounded interval  $[a, b]$  to be Riemann integrable.  
 b) Using your definition from part (a), prove that any monotonically increasing function on  $[0, 1]$  is Riemann integrable.

62. Given the vector field  $\mathbf{V}(x, y, z) = (4y, x, 2z)$  in 3-space, find the value of the integral

$$\iint_H \text{curl } \mathbf{V} \cdot \mathbf{n} dA$$

where  $H$  is the hemisphere  $x^2 + y^2 + z^2 = a^2$ ,  $z \geq 0$ ,  $\mathbf{n}$  is the unit outward normal, and  $dA$  is the element of area.

63. a) Let  $c(x)$  be a given smooth function and  $u(x) \not\equiv 0$  satisfy the differential equation  $-u'' + c(x)u = \lambda u$  on the bounded interval  $\Omega = \{a < x < b\}$  with  $u = 0$  on the boundary of  $\Omega$ . Show that

$$\lambda = \frac{\int_{\Omega} (u'^2 + cu^2) dx}{\int_{\Omega} u^2 dx}$$

- b) Let  $c(x, y)$  be a given smooth function and  $u(x, y) \not\equiv 0$  satisfy the differential equation  $-(u_{xx} + u_{yy}) + cu = \lambda u$  on a bounded set  $\Omega \subset \mathbb{R}^2$  with  $u = 0$  on the boundary of  $\Omega$ . Show that

$$\lambda = \frac{\iint_{\Omega} (|\nabla u|^2 + cu^2) dx dy}{\iint_{\Omega} u^2 dx dy}$$

64. Investigate the continuity and differentiability of

$$f(x) = \begin{cases} |x|^p \cos \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

where  $p$  is a real number.

65. Determine the radius of convergence of the series  $\sum_{n=0}^{\infty} \frac{x^{n^2}}{n!}$ .
66. Calculate  $\lim_{x \rightarrow \infty} \{x^{1/3}[(x+1)^{2/3} - (x-1)^{2/3}]\}$ .
67. Prove that the function  $f(x) = \left(1 + \frac{1}{x}\right)^x$  is monotone increasing for  $x > 0$ .
68. Compute  $\int \frac{dx}{\sin x + \cos x}$ .
69. Find the critical points of each of the following functions defined on the plane  $\mathbb{R}^2$ . Also, where possible, classify these critical points as local maxima, minima, or saddles.
- $f(x, y) = x^4 + y^4 - 4xy + 1$
  - $g(x, y) = x^2y^2$
  - $\frac{\cos x}{1 + y^2}$
70. Find an example of a smooth function  $f(x, y)$  defined on the whole plane  $\mathbb{R}^2$  that has exactly three critical points, all non-degenerate, with one a local maximum, one a local minimum, and the third a saddle point.
71. Here we use the series  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$  to show that  $e$  is irrational.
- Show that  $2 < e < 3$ , so  $e$  is not an integer.
  - Assume  $e = p/q$  is rational, with  $p$  and  $q$  integers with  $q \geq 2$ . Use Taylor series with  $q$  terms and remainder  $R_q$  to show that  $e \cdot q! = N + \frac{e^c}{q+1}$ , where  $N$  is an integer and  $0 < c < 1$ .
  - Deduce that  $\frac{e^c}{q+1}$  is an integer. Show this contradicts  $e^c < e^1 < 3$  and  $q+1 \geq 3$ .
72. Let  $h(x) \geq 0$  be a continuous monotonically decreasing function for  $0 \leq x \leq \infty$  with the property that  $\lim_{x \rightarrow \infty} h(x) = 0$ . Show that the improper integral  $\int_0^{\infty} h(x) \sin x dx$  exists.
73. Let  $f \in C^2(a, b)$  and say  $x_0 \in (a, b)$ . If  $f''(x_0) > 0$ , give an analytic proof that near  $x_0$  the graph of  $y = f(x)$  lies above its tangent line at  $(x_0, f(x_0))$ .

74. Let  $I_k = \int_{-\infty}^{\infty} x^{2k} e^{-x^2} dx$ ,  $k \geq 0$ .

a) Show that  $I_k = \frac{2k-1}{2} I_{k-2}$ .

b) Compute  $I_2, I_3, I_4, I_5, I_6$ , and  $I_7$ . [You may use that  $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ ].

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