

Calculus Problems

Math 504 – 505

Jerry L. Kazdan

1. A certain function $f(x)$ has the property that $\int_0^x f(t) dt = e^x \cos x + C$. Find both f and the constant C .
2. Compute $\lim_{x \rightarrow 0} \left(\frac{\cos x}{\cos 2x} \right)^{1/x^2}$.
3. Sketch the curve that is defined implicitly by $x^3 + y^3 - 3xy = 0$. Calculate $y'(0)$.
4. Calculate $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.
5. Determine the indefinite integral $\int \log(1 + x^2) dx$.
6. Let $f(x)$ be a smooth function for $0 \leq x \leq 1$. If $f'(x) = 0$ for all $0 \leq x \leq 1$, what can you conclude? Prove all your assertions.
7. Solve the initial value problem $(1 + e^x)yy' = e^x$ with $y(1) = 1$.
8. Let the continuous function $f(\theta)$, $0 \leq \theta \leq 2\pi$ represent the temperature along the equator at a certain moment, say measured from the longitude at Greenwich.. Show there are antipodal points with the *same* temperature.
9. a) Let $g(x) := x^3(1 - x)$. Without computation, show that $g'''(c) = 0$ for some $0 < c < 1$.
b) Let $h(x) := x^3(1 - x)^3$. Show that $h'''(x)$ has exactly three distinct roots in the interval $0 < x < 1$.
c) Let $p(x) := \left(\frac{d}{dx} \right)^4 (1 - x^2)^4$. Show that p is a polynomial of degree 4 and that it has 4 real distinct zeroes, all lying in the interval $-1 < x < 1$.
10. a) If $f(x) > 0$ is continuous for all $x \geq 0$ and the improper integral $\int_0^{\infty} f(x) dx$ exists, then $\lim_{x \rightarrow \infty} f(x) = 0$. Proof or counterexample.

- b) If $f(x) > 0$ is continuous for all $x \geq 0$ and the improper integral $\int_0^\infty f(x) dx$ exists, then $f(x)$ is bounded. Proof or counterexample.
11. Find *explicit* rational functions $f(x)$ and $g(x)$ with the following Taylor series: $f(x) = \sum_1^\infty nx^n$, $g(x) = \sum_1^\infty n^2x^n$.
12. a) Let x be a point in \mathbb{R}^2 and consider $\int_{\mathbb{R}^2} \frac{1}{(1 + \|x\|^2)^p} dx$. For which p does this improper integral converge?
 b) This integral can be computed explicitly. Do so.
 c) Repeat part a) where $x \in \mathbb{R}^3$ and the integral is over \mathbb{R}^3 instead of \mathbb{R}^2 .
13. Compute $\iint_{\mathbb{R}^2} \frac{1}{[1 + (2x + y + 1)^2 + (x - y + 3)^2]^2} dx dy$.
14. Let $v(x, t) := \int_{x-2t}^{x+2t} g(s) ds$, where g is a continuous function. Compute $\partial v / \partial t$ and $\partial v / \partial x$.
15. Let $H(t) := \int_{a(t)}^{b(t)} f(x, t) dx$, where $a(t)$, $b(t)$, and $f(x, t)$ are smooth functions of their variables. Compute dH/dt .
16. a) Let $p(x) := x^3 + cx + d$, where c , and d are real. Under what conditions on c and d does this has three distinct real roots? [ANSWER: $c < 0$ and $d^2 < -4c^3/27$].
 b) Generalize to the real polynomial $p(x) := ax^3 + bx^2 + cx + d$ ($a \neq 0$) by a change of variable reducing to the above special case.
17. If $b \geq 0$, show that for every real c the equation $x^5 + bx + c = 0$ has exactly one real root. What if $b < 0$? Say as much as you can.
18. Let $f(t)$ and $g(t)$ be smooth increasing functions of $t \in \mathbb{R}$. Proof or counterexample:
 a) $f(t) + g(t)$ is an increasing functions of t .
 b) $f(t)g(t)$ is an increasing functions of t .
 c) If $f(t) > 0$ and $g(t) > 1$ then $f(t)^{g(t)}$ is an increasing functions of t .

19. Let a smooth function $g(x)$ have the three properties: $g(0) = 2$, $g(1) = 0$, $g(4) = 6$. Show that at some point $0 < c < 4$ one has $g''(c) > 0$. Better yet, find a number $m > 0$ so that $g''(c) \geq m > 0$.

Is it true that g'' must be positive at at least one point $0 < c < 1$? Proof or counterexample.

20. Let $f(x)$ be a differentiable function for all real x with the property that $f'(x) < 1$ for all x . Show has at most one *fixed point*, that is, at most one point c where $f(c) = c$.

21. Let g be a differentiable function with the properties $g(a) = 0$, $g(b) = 0$, and $g'(x) \geq 0$ for all $x \in [a, b]$. What can you deduce about $g(x)$ for $x \in [a, b]$? Justify your conclusions.

22. Let $v(x)$ be a smooth real-valued function for $0 \leq x \leq 1$. If $v(0) = v(1) = 0$ and $v''(x) > 0$ for all $0 \leq x \leq 1$, show that $v(x) \leq 0$ for all $0 \leq x \leq 1$.

23. If a smooth curve $y = f(x)$ has the property that $f''(x) > 0$, show that the chord joining two points of the curve lies above the curve:

$$tf(b) + (1-t)f(a) \geq f(tb + (1-t)a) \quad \text{for all } 0 \leq t \leq 1.$$

24. a) Find a number N so that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{N}} > 100$.

b) Find an integer N so that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} > 100$.

25. Let c be any complex number. Show that $\lim_{n \rightarrow \infty} \frac{c^n}{n!} = 0$.

26. a) Show that $\sin x$ is not a polynomial.

b) Show that $\sin x$ is not a rational function, that is, it cannot be the quotient of two polynomials.

c) Let $f(t)$ be periodic with period 1, so $f(t+1) = f(t)$ for all real t . If f is not a constant, show that it cannot be a rational function. that is, f cannot be the quotient of two polynomials.

d) Show that e^x is not a rational function.

27. Let $f(x)$ be a differentiable function of $x := (x_1, x_2, x_3)$ for all $x \in \mathbb{R}^3$. If f is *homogeneous of degree k* in the sense that $f(cx) = c^k f(x)$ for all $c > 0$, show that $x \cdot \nabla f(x) = kf(x)$ (Euler).
28. The *Gamma function* is defined by $\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt$.
- For which real x does this improper integral converge?
 - Show that $\Gamma(x+1) = x\Gamma(x)$ and deduce that $\Gamma(n+1) = n!$ for any integer $n \geq 0$.
29. Say $\gamma(t) : \mathbb{R} \rightarrow \mathbb{R}^2$ defines a smooth curve in the plane.
- If $\gamma(0) = 0$ and $\|\gamma'(t)\| \leq c$, show that for any $T \geq 0$, $\|\gamma(T)\| \leq cT$. Moreover, show that equality can occur if and only if one has $\gamma(t) = cvt$ where v is a unit vector that does not depend on t .
 - If $\gamma(0) = 0$, $\gamma'(0) = 0$ and $\|\gamma''(t)\| \leq 12$, give an upper bound estimate for $\|\gamma(2)\|$. When can this upper bound be achieved?
30. Let $\mathbf{r}(t)$ define a smooth curve that does not pass through the origin.
- If the point $\mathbf{a} = \mathbf{r}(t_0)$ is a point on the curve that is closest to the origin (and *not* an end point of the curve), show that the position vector $\mathbf{r}(t_0)$ is perpendicular to the tangent vector $\mathbf{r}'(t_0)$.
 - What can you say about a point $\mathbf{b} = \mathbf{r}(t_1)$ that is *furthest* from the origin?
31. Consider two smooth plane curves $\gamma_1, \gamma_2 : (0, 1) \rightarrow \mathbb{R}^2$ that do not intersect. Suppose P_1 and P_2 are points on γ_1 and γ_2 , respectively, such that the distance $|P_1P_2|$ is minimal. Prove that the straight line P_1P_2 is normal to *both* curves.
32. Let $h(x, y, z) = 0$ define a smooth surface in \mathbb{R}^3 and let $P := (a, b, c)$ be a point *not* on the surface. If $Q := (x, y, z)$ is a point on the surface that is closest to P , show that the line PQ is perpendicular to the tangent plane to the surface at Q .
33. Let $\mathbf{r}(t)$ describe a smooth curve and let \mathbf{V} be a fixed vector. If $\mathbf{r}'(t)$ is perpendicular to \mathbf{V} for all t and if $\mathbf{r}(0)$ is perpendicular to \mathbf{V} , show that $\mathbf{r}(t)$ is perpendicular to \mathbf{V} for all t .
34. Let $f(s)$ be any differentiable function of the real variable s . Show that $u(x, t) := f(x+3t)$ has the property that $u_t = 3u_x$. Show that u also satisfies the wave equation $u_{tt} = 9u_{xx}$.

35. Let $u(x, y)$ be a smooth function.
- If $u_x = 0$ with $u(0, y) = \sin(3y)$, find $u(x, y)$.
 - If $u_x = 2xy$ with $u(0, y) = \sin(3y)$, find $u(x, y)$.
 - If $u_x + u_y = 0$ with $u(0, y) = \sin(3y)$, find $u(x, y)$. Is there more than one such function?
 - If $u_x + u_y = 3 - 2xy$ with $u(0, y) = \sin(3y)$, find $u(x, y)$. Is there more than one such function?
 - If $u_x - 2u_y = 0$ with $u(0, y) = \sin(3y)$, find $u(x, y)$. Is there more than one such function?
36. Let $\mathbf{r} := x\mathbf{i} + y\mathbf{j}$ and $\mathbf{V}(x, y) := p(x, y)\mathbf{i} + q(x, y)\mathbf{j}$ be (smooth) vector fields and C a smooth curve in the plane. In this problem I is the line integral $I = \int_C \mathbf{V} \cdot d\mathbf{r}$. For each of the following, either give a proof or give a counterexample.
- If C is a vertical line segment and $q(x, y) = 0$, then $I = 0$.
 - If C is a circle and $q(x, y) = 0$, then $I = 0$.
 - If C is a circle centered at the origin and $p(x, y) = -q(x, y)$, then $I = 0$.
 - If $p(x, y) > 0$ and $q(x, y) > 0$, then $I > 0$.
37. Let C denote the unit circle centered at the origin of the plane, and D denote the circle of radius 5 centered at $(2, 1)$, both oriented counterclockwise. Let Q denote the ring region between these curves. If a vector field \mathbf{V} satisfies $\operatorname{div} \mathbf{V} = 0$, show that the line integral $\int_C \mathbf{V} \cdot \mathbf{N} ds = \int_D \mathbf{V} \cdot \mathbf{N} ds = 0$. [This extends immediately to the situation where C and D are more general curves and Q is the region between them. For fluid flow it is an expression of *conservation of mass*, since $\operatorname{div} \mathbf{V} = 0$ means there are no sources or sinks in the region Q .]
38. (Integration by Parts for Multiple Integrals) Let \mathbf{F} be a smooth vector field and u a smooth scalar-valued function.
- Prove the identity $\nabla \cdot (u\mathbf{F}) = \nabla u \cdot \mathbf{F} + u\nabla \cdot \mathbf{F}$. Compare this with the special case of a function of one variable.
 - Let \mathcal{D} be a bounded region in the plane whose boundary is the curve C with unit outer normal \mathbf{N} . Also, let u be a scalar-valued function, and \mathbf{F} a vector field. Prove the identity

$$\iint_{\mathcal{D}} u \nabla \cdot \mathbf{F} dA = \oint_C u \mathbf{F} \cdot \mathbf{N} ds - \iint_{\mathcal{D}} \nabla u \cdot \mathbf{F} dA.$$

Notice that for a function of one variable with \mathcal{D} being the interval $\{a < x < b\}$, this reduces precisely to the usual formula for integration by parts.

- c) Generalize this formula to the case where \mathcal{D} is a bounded (solid) region in three dimensional space.
- d) One frequently uses this with $\mathbf{F} = \nabla v$. Show the above formula for integration by parts becomes (say in two dimensions)

$$\iint_{\mathcal{D}} u \nabla \cdot \nabla v dA = \oint_C u \nabla v \cdot \mathbf{N} ds - \iint_{\mathcal{D}} \nabla u \cdot \nabla v dA.$$

This is *Green's theorem*. To what does this reduce for functions on one variable?

- e) As a short application using this, say $u(x, y)$ is a *harmonic function* in a bounded region \mathcal{D} , so $\Delta u := \nabla \cdot \nabla u = u_{xx} + u_{yy} = 0$. One can think of $u(x, y)$ as being the equilibrium temperature of \mathcal{D} . Let C is the boundary of \mathcal{D} . If $u = 0$ on C , it is plausible that one must have $u(x, y) = 0$ throughout \mathcal{D} . Show how this follows from the above formula. What is the analogous assertion for functions of one variable, where a harmonic function is just a solution of $u'' = 0$?

39. Let \mathcal{D} be a bounded region in the plane, and let \mathcal{B} be its boundary.

- a) Use the divergence theorem (or any related formula you know) to show that for any smooth function $v(x, y)$

$$\iint_{\mathcal{D}} \Delta v dx dy = \int_{\mathcal{B}} \frac{\partial v}{\partial N} ds$$

where $\partial v / \partial N := \nabla v \cdot \mathbf{n}$ is the outer normal directional derivative on \mathcal{B} .

- b) Let $u(x, y, t)$ be a solution of the heat equation $u_t = \Delta u$ for (x, y) in \mathcal{D} . Assume that the boundary, \mathcal{B} , is *insulated*, so the outer normal derivative there is zero: $\frac{\partial u}{\partial N} = 0$ on \mathcal{B} .

Show that $Q(t) := \iint_{\mathcal{D}} u(x, y, t) dx dy$ is a constant.

40. Continuing the notation of the previous problem, say that instead the temperature $u(x, y, t) = 0$ for all points (x, y) on the boundary \mathcal{B} .

- a) Show that the function $E(t) := \frac{1}{2} \int_{\mathcal{D}} u^2(x, y, t) dx dy$ has the property that $dE/dt \leq 0$.
- b) Use this to show that with these zero boundary conditions, if the initial temperature is zero, $u(x, y, 0) = 0$, then $u(x, y, t) = 0$ for all $t \geq 0$.

41. Let $u(x, y, t)$ describe the motion of a vibrating drumhead \mathcal{D} . A reasonable mathematical model shows that u satisfies the *wave equation* $u_{tt} = \Delta u$ in \mathcal{D} with *boundary condition* $u(x, y, t) = 0$ for all (x, y) on the boundary $\partial \mathcal{D}$.

Physical reasoning leads one to define the *energy* as

$$E(t) := \frac{1}{2} \iint_{\mathcal{D}} (u_t^2 + |\nabla u|^2) dA.$$

- a) Show that energy is conserved: $E(t) = E(0)$. [HINT: Show $dE/dt = 0$.]
- b) If in addition one knows that the initial position $u(x, y, 0) = 0$ and that the *initial velocity* $u_t(x, y, 0) = 0$, show that $E(t) = 0$ for all t and deduce that $u(x, y, t) \equiv 0$. [This is hardly a surprise on physical grounds, but it should be interpreted as reassuring us that this mathematical model is indeed the correct model.]

42. If $h'(t) \leq ch(t)$, where c is a constant, show that $h(t) \leq e^{ct}h(0)$ for all $t \geq 0$.
43. Say $u(t)$ satisfies $u'' + b(t)u' + c(t)u = 0$, where $b(t)$ and $c(t)$ are bounded functions. Let $E(t) := \frac{1}{2}(u'^2 + u^2)$.
- a) Show that $E'(t) \leq \gamma E(t)$, where γ is a constant.
 - b) Deduce that if $u(0) = 0$ and $u'(0) = 0$, then $u(t) = 0$ for all t .
44. Let $\mathcal{V} := \{u(x) \in C^2(\mathbb{R}) \mid u'' + u = 0\}$. Prove that $\dim \mathcal{V} = 2$. Prove all of your assertions in detail.

45. The solutions to the following matrix differential equation

$$X' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} X$$

form a vector space. Find a basis for this vector space.

46. Consider the differential equation $X'(t) = AX(t)$ where

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Which of the following assertions are correct—and why?

- a) There is a solution of the form $X(t) = U$, where U is a real constant (non-zero) vector.

- b) There is a solution of the form $X(t) = Ve^{2t}$, where V is a real constant (non-zero) vector.
- c) There is a solution of the form $X(t) = Ve^{-2t}$, where V is a real constant (non-zero) vector.
- d) There is a complex solution of the form $X(t) = We^{it}$, where W is a constant (non-zero) vector.
- e) All of the solutions of this equation remain bounded as $t \rightarrow \infty$.
47. Consider the *second order* differential equation $X'' = AX$ where A is a symmetric 2×2 matrix.
- a) Find the general solution if $A = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$.
- b) Find the general solution if $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$. [Suggestion: First diagonalize A , so $D := R^{-1}AR$ is diagonal. Then make the change of variables $X = RY$ to obtain a simpler differential equation for $Y(t)$.]
- c) Find the general solution if $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.
48. For which complex numbers z does the series $\sum_1^\infty ne^{-nz}$ converge?
49. a) Let $u(x_1, \dots, x_n)$ be a smooth function that depends only on the distance $r = \sqrt{x_1^2 + \dots + x_n^2}$. Show that
- $$\frac{\partial^2 u}{\partial x_j^2} = \frac{x_j^2}{r^2} \frac{d^2 u}{dr^2} + \frac{(r^2 - x_j^2)}{r^3} \frac{du}{dr}, \quad \text{and hence} \quad \Delta u = u_{rr} + \frac{n-1}{r} u_r.$$
- b) Find all harmonic functions (these are the solutions of $\Delta u = 0$) that depend only on r .
50. a) Find the equation of the tangent plane to the surface $x^2 + xy + y^3 - z^2 = 2$ at the point $(1, 1, 1)$.
- b) Say the function $T = x^2 + xy + y^3 - z^2$ gives the temperature at the point (x, y, z) . At the point $(1, 1, 1)$, in which direction should one move so that the temperature increases fastest?

51. Let $\psi(t)$ be a scalar-valued function with a continuous derivative for $0 < t < \infty$ and let $\mathbf{X} = (x, y, z) \in \mathbb{R}^3$. Define the vector field $\mathbf{F}(\mathbf{X}) := \psi(\|\mathbf{X}\|)\mathbf{X}$ for all $\mathbf{X} \neq \mathbf{0}$. Show that this vector field is conservative by finding a scalar-valued function $\phi(r)$ with the property that $\mathbf{F}(\mathbf{X}) := \nabla\phi(\|\mathbf{X}\|)$. In particular, this shows that *every central force field is conservative*.

52. Let \mathcal{D} be a bounded region in the plane with smooth boundary \mathcal{B} . Show that

$$\text{Area}(\mathcal{D}) = \frac{1}{2} \int_{\mathcal{B}} xdy - ydx.$$

Use this to find the area inside the ellipse $(x, y) = (a \cos \theta, b \sin \theta)$ for $0 \leq \theta \leq 2\pi$.

53. If $\{b_j\} > 0$, prove the arithmetic-geometric mean inequality

$$(b_1 b_2 \cdots b_n)^{1/n} \leq \frac{b_1 + b_2 + \cdots + b_n}{n}.$$

When is there equality?

54. Let $0 < c < 1$. Show that $s^c t^{1-c} < cs + (1-c)t$ for all $s, t > 0$, $s \neq t$ (if $s = t$, then this becomes an equality).

55. Let $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$ for all $x, y > 0$.

56. Let P_1, \dots, P_n be $n \geq 3$ points on a circle and let Q be the polygon obtained by connecting these successive points. How should the points be situated to maximize the area of Q ?

57. a) Find a smooth function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ that has exactly three critical points, all non-degenerate, one being a local max, one a local min, and the third a saddle.

b) Show there is no such $f(x, y)$ of the form $f(x, y) = g(x) + h(y)$.

58. Compute $\lim_{\lambda \rightarrow \infty} \int_0^1 |\sin \lambda x| dx$ (part of the problem is to show that the limit exists).

59. a) State what it means for a real-valued function defined on the closed, bounded interval $[a, b]$ to be Riemann integrable.

b) Using your definition from part (a), prove that any monotonically increasing function on $[0,1]$ is Riemann integrable.

60. Given the vector field $\mathbf{V}(x,y,z) = (4y,x,2z)$ in 3-space, find the value of the integral

$$\iint_H \text{curl } \mathbf{V} \cdot \mathbf{n} \, dA$$

where H is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, \mathbf{n} is the unit outward normal, and dA is the element of area.

61. a) Let $c(x)$ be a given smooth function and $u(x) \not\equiv 0$ satisfy the differential equation $-u'' + c(x)u = \lambda u$ on the bounded interval $\Omega = \{a < x < b\}$ with $u = 0$ on the boundary of Ω . Show that

$$\lambda = \frac{\int_{\Omega} (u'^2 + cu^2) \, dx}{\int_{\Omega} u^2 \, dx}$$

b) Let $c(x,y)$ be a given smooth function and $u(x,y) \not\equiv 0$ satisfy the differential equation $-(u_{xx} + u_{yy}) + cu = \lambda u$ on a bounded set $\Omega \subset \mathbb{R}^2$ with $u = 0$ on the boundary of Ω . Show that

$$\lambda = \frac{\iint_{\Omega} (|\nabla u|^2 + cu^2) \, dx \, dy}{\iint_{\Omega} u^2 \, dx \, dy}$$

62. Investigate the continuity and differentiability of

$$f(x) = \begin{cases} |x|^p \cos \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

where p is a real number.

63. Determine the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^{n^2}}{n!}$.

64. Calculate $\lim_{x \rightarrow \infty} \{x^{1/3} [(x+1)^{2/3} - (x-1)^{2/3}]\}$.

65. Prove that the function $f(x) = \left(1 + \frac{1}{x}\right)^x$ is monotone increasing for $x > 0$.

66. Compute $\int \frac{dx}{\sin x + \cos x}$.
67. Find the critical points of each of the following functions defined on the plane \mathbb{R}^2 . Also, where possible, classify these critical points as local maxima, minima, or saddles.
- $f(x, y) = x^4 + y^4 - 4xy + 1$
 - $g(x, y) = x^2y^2$
 - $\frac{\cos x}{1 + y^2}$
68. Find an example of a smooth function $f(x, y)$ defined on the whole plane \mathbb{R}^2 that has exactly three critical points, all non-degenerate, with one a local maximum, one a local minimum, and the third a saddle point.
69. Here we use the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ to show that e is irrational.
- Show that $2 < e < 3$, so e is not an integer.
 - Assume $e = p/q$ is rational, with p and q integers with $q \geq 2$. Use Taylor series with q terms and remainder R_q to show that $e \cdot q! = N + \frac{e^c}{q+1}$, where N is an integer and $0 < c < 1$.
 - Deduce that $\frac{e^c}{q+1}$ is an integer. Show this contradicts $e^c < e^1 < 3$ and $q + 1 \geq 3$.
70. Let $h(x) \geq 0$ be a continuous monotonically decreasing function for $0 \leq x \leq \infty$ with the property that $\lim_{x \rightarrow \infty} h(x) = 0$. Show that the improper integral $\int_0^\infty h(x) \sin x dx$ exists.
71. Let $f \in C^2(a, b)$ and say $x_0 \in (a, b)$. If $f''(x_0) > 0$, give an analytic proof that near x_0 the graph of $y = f(x)$ lies above its tangent line at $(x_0, f(x_0))$.
72. Let $I_k = \int_{-\infty}^{\infty} x^{2k} e^{-x^2} dx$, $k \geq 0$.
- Show that $I_k = \frac{2k-1}{2} I_{k-2}$.
 - Compute I_2, I_3, I_4, I_5, I_6 , and I_7 . [You may use that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$].

[Last revised: November 14, 2006]