

# Linear Algebra Problems

Math 504 – 505

Jerry L. Kazdan

Although problems are categorized by topics, this should not be taken very seriously since many problems fit equally well in several different topics.

NOTATION: We occasionally write  $M(n, \mathbb{F})$  for the ring of all  $n \times n$  matrices over the field  $\mathbb{F}$ , where  $\mathbb{F}$  is either  $\mathbb{R}$  or  $\mathbb{C}$ .

## Basics

- At noon the minute and hour hands of a clock coincide.
  - What in the first time,  $T_1$ , when they are perpendicular?
  - What is the next time,  $T_2$ , when they again coincide?
- Which of the following sets are linear spaces?
  - $\{X = (x_1, x_2, x_3) \text{ in } \mathbb{R}^3 \text{ with the property } x_1 - 2x_3 = 0\}$
  - The set of solutions  $x$  of  $Ax = 0$ , where  $A$  is an  $m \times n$  matrix.
  - The set of  $2 \times 2$  matrices  $A$  with  $\det(A) = 0$ .
  - The set of polynomials  $p(x)$  with  $\int_{-1}^1 p(x) dx = 0$ .
  - The set of solutions  $y = y(t)$  of  $y'' + 4y' + y = 0$ .
- Which of the following sets of vectors are bases for  $\mathbb{R}^2$ ?

a). $\{(0, 1), (1, 1)\}$	d). $\{(1, 1), (1, -1)\}$
b). $\{(1, 0), (0, 1), (1, 1)\}$	e). $\{((1, 1), (2, 2)\}$
c). $\{(1, 0), (-1, 0)\}$	f). $\{(1, 2)\}$
- For which real numbers  $x$  do the vectors:  $(x, 1, 1, 1)$ ,  $(1, x, 1, 1)$ ,  $(1, 1, x, 1)$ ,  $(1, 1, 1, x)$  *not* form a basis of  $\mathbb{R}^4$ ? For each of the values of  $x$  that you find, what is the dimension of the subspace of  $\mathbb{R}^4$  that they span?
- Let  $C(\mathbb{R})$  be the linear space of all continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ .
  - Let  $S_c$  be the set of differentiable functions  $u(x)$  that satisfy the differential equation
$$u' = 2xu + c$$
for all real  $x$ . For which value(s) of the real constant  $c$  is this set a linear subspace of  $C(\mathbb{R})$ ?

- b) Let  $C^2(\mathbb{R})$  be the linear space of all functions from  $\mathbb{R}$  to  $\mathbb{R}$  that have two continuous derivatives and let  $S_f$  be the set of solutions  $u(x) \in C^2(\mathbb{R})$  of the differential equation

$$u'' + u = f(x)$$

for all real  $x$ . For which polynomials  $f(x)$  is the set  $S_f$  a linear subspace of  $C(\mathbb{R})$ ?

- c) Let  $\mathcal{A}$  and  $\mathcal{B}$  be linear spaces and  $L : \mathcal{A} \rightarrow \mathcal{B}$  be a linear map. For which vectors  $y \in \mathcal{B}$  is the set

$$\mathcal{S}_y := \{x \in \mathcal{A} \mid Lx = y\}$$

a linear space?

6. Compute the dimension and find bases for the following linear spaces.

- Real anti-symmetric  $4 \times 4$  matrices.
- Quartic polynomials  $p$  with the property that  $p(2) = 0$  and  $p(3) = 0$ .
- Cubic polynomials  $p(x, y)$  in two real variables with the properties:  $p(0, 0) = 0$ ,  $p(1, 0) = 0$  and  $p(0, 1) = 0$ .
- The space of linear maps  $L : \mathbb{R}^5 \rightarrow \mathbb{R}^3$  whose kernels contain  $(0, 2, -3, 0, 1)$ .

7. a) Compute the dimension of the intersection of the following two planes in  $\mathbb{R}^3$

$$x + 2y - z = 0, \quad 3x - 3y + z = 0.$$

- b) A map  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is defined by the matrix  $L := \begin{pmatrix} 1 & 1 & -1 \\ 3 & -3 & 1 \end{pmatrix}$ . Find the nullspace (kernel) of  $L$ .

8. If  $A$  is a  $5 \times 5$  matrix with  $\det A = -1$ , compute  $\det(-2A)$ .

9. Does an 8-dimensional vector space contain linear subspaces  $V_1, V_2, V_3$  with no common non-zero element, such that

$$\text{a). } \dim(V_i) = 5, \quad i = 1, 2, 3? \quad \text{b). } \dim(V_i) = 6, \quad i = 1, 2, 3?$$

10. Let  $U$  and  $V$  both be two-dimensional subspaces of  $\mathbb{R}^5$ , and let  $W = U \cap V$ . Find all possible values for the dimension of  $W$ .

11. Let  $U$  and  $V$  both be two-dimensional subspaces of  $\mathbb{R}^5$ , and define the set  $W := U + V$  as the set of all vectors  $w = u + v$  where  $u \in U$  and  $v \in V$  can be any vectors.

- a) Show that  $W$  is a linear space.

- b) Find all possible values for the dimension of  $W$ .
12. Let  $A$  be an  $n \times n$  matrix of real or complex numbers. Which of the following statements are *equivalent* to: “the matrix  $A$  is invertible”?
- The columns of  $A$  are linearly independent.
  - The columns of  $A$  span  $\mathbb{R}^n$ .
  - The rows of  $A$  are linearly independent.
  - The kernel of  $A$  is  $0$ .
  - The only solution of the homogeneous equations  $Ax = 0$  is  $x = 0$ .
  - The linear transformation  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by  $A$  is 1-1.
  - The linear transformation  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by  $A$  is onto.
  - The rank of  $A$  is  $n$ .
  - The adjoint,  $A^*$ , is invertible.
  - $\det A \neq 0$ .
13. Call a subset  $S$  of a vector space  $V$  a *spanning set* if  $\text{Span}(S) = V$ . Suppose that  $T : V \rightarrow W$  is a linear map of vector spaces.
- Prove that a linear map  $T$  is 1-1 if and only if  $T$  sends linearly independent sets to linearly independent sets.
  - Prove that  $T$  is onto if and only if  $T$  sends spanning sets to spanning sets.

### Linear Equations

14. Say you have  $k$  linear algebraic equations in  $n$  variables; in matrix form we write  $AX = Y$ . Give a proof or counterexample for each of the following.
- If  $n = k$  there is always *at most one* solution.
  - If  $n > k$  you can *always* solve  $AX = Y$ .
  - If  $n > k$  the nullspace of  $A$  has dimension greater than zero.
  - If  $n < k$  then for *some*  $Y$  there is *no* solution of  $AX = Y$ .
  - If  $n < k$  the *only* solution of  $AX = 0$  is  $X = 0$ .
15. Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be a linear map. Show that the following are equivalent.
- $A$  is injective (hence  $n \leq k$ ).
  - $\dim \ker(A) = 0$ .

- c)  $A$  has a *left inverse*  $B$ , so  $BA = I$ .
- d) The columns of  $A$  are linearly independent.
16. Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be a linear map. Show that the following are equivalent.
- $A$  is surjective (hence  $n \geq k$ ).
  - $\dim \operatorname{im}(A) = k$ .
  - $A$  has a *right inverse*  $B$ , so  $AB = I$ .
  - The columns of  $A$  span  $\mathbb{R}^k$ .
17. Let  $A$  be a  $4 \times 4$  matrix with determinant 7. Give a proof or counterexample for each of the following.
- For some vector  $\mathbf{b}$  the equation  $A\mathbf{x} = \mathbf{b}$  has exactly one solution.
  - some vector  $\mathbf{b}$  the equation  $A\mathbf{x} = \mathbf{b}$  has infinitely many solutions.
  - For some vector  $\mathbf{b}$  the equation  $A\mathbf{x} = \mathbf{b}$  has no solution.
  - For all vectors  $\mathbf{b}$  the equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution.
18. Let  $A$  and  $B$  be  $n \times n$  matrices with  $AB = 0$ . Give a proof or counterexample for each of the following.
- Either  $A = 0$  or  $B = 0$  (or both).
  - $BA = 0$
  - If  $\det A = -3$ , then  $B = 0$ .
  - If  $B$  is invertible then  $A = 0$ .
  - There is a vector  $V \neq 0$  such that  $BAV = 0$ .
19. Consider the system of equations

$$\begin{aligned}x + y - z &= a \\x - y + 2z &= b.\end{aligned}$$

- Find the general solution of the homogeneous equation.
- A particular solution of the inhomogeneous equations when  $a = 1$  and  $b = 2$  is  $x = 1, y = 1, z = 1$ . Find the most general solution of the inhomogeneous equations.
- Find some particular solution of the inhomogeneous equations when  $a = -1$  and  $b = -2$ .

- d) Find some particular solution of the inhomogeneous equations when  $a = 3$  and  $b = 6$ .

[Remark: After you have done part a), it is possible immediately to write the solutions to the remaining parts.]

20. Let  $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ .

- a) Find the general solution  $\mathbf{Z}$  of the homogeneous equation  $A\mathbf{Z} = 0$ .
- b) Find some solution of  $A\mathbf{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- c) Find the general solution of the equation in part b).
- d) Find some solution of  $A\mathbf{X} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$  and of  $A\mathbf{X} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$
- e) Find some solution of  $A\mathbf{X} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$
- f) Find some solution of  $A\mathbf{X} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ . [Note:  $\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ ].

[Remark: After you have done parts a), b) and e), it is possible immediately to write the solutions to the remaining parts.]

21. Consider the system of equations

$$\begin{aligned}x + y - z &= a \\x - y + 2z &= b \\3x + y &= c\end{aligned}$$

- a) Find the general solution of the homogeneous equation.
- b) If  $a = 1$ ,  $b = 2$ , and  $c = 4$ , then a particular solution of the inhomogeneous equations is  $x = 1$ ,  $y = 1$ ,  $z = 1$ . Find the most general solution of these inhomogeneous equations.
- c) If  $a = 1$ ,  $b = 2$ , and  $c = 3$ , show these equations have *no* solution.
- d) If  $a = 0$ ,  $b = 0$ ,  $c = 1$ , show the equations have *no* solution. [Note:  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ].
- e) Let  $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$ . Compute  $\det A$ .

[Remark: After you have done parts a), and c), it is possible immediately to write the solutions to the remaining parts with no additional computation.]

## Linear Maps

22. a) Find a  $2 \times 2$  matrix that rotates the plane by  $+45$  degrees ( $+45$  degrees means 45 degrees *counterclockwise*).
- b) Find a  $2 \times 2$  matrix that rotates the plane by  $+45$  degrees followed by a reflection across the horizontal axis.
- c) Find a  $2 \times 2$  matrix that reflects across the horizontal axis followed by a rotation the plane by  $+45$  degrees.
- d) Find a matrix that rotates the plane through  $+60$  degrees, keeping the origin fixed.
- e) Find the inverse of each of these maps.
23. a) Find a  $3 \times 3$  matrix that acts on  $\mathbb{R}^3$  as follows: it keeps the  $x_1$  axis fixed but rotates the  $x_2$   $x_3$  plane by 60 degrees.
- b) Find a  $3 \times 3$  matrix  $A$  mapping  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$  that rotates the  $x_1$   $x_3$  plane by 60 degrees and leaves the  $x_2$  axis fixed.
24. Find a real  $2 \times 2$  matrix  $A$  (other than  $A = I$ ) such that  $A^5 = I$ .
25. Proof or counterexample. In these  $L$  is a linear map from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , so its representation will be as a  $2 \times 2$  matrix.
- a) If  $L$  is invertible, then  $L^{-1}$  is also invertible.
- b) If  $LV = 5V$  for all vectors  $V$ , then  $L^{-1}W = (1/5)W$  for all vectors  $W$ .
- c) If  $L$  is a rotation of the plane by 45 degrees *counterclockwise*, then  $L^{-1}$  is a rotation by 45 degrees *clockwise*.
- d) If  $L$  is a rotation of the plane by 45 degrees counterclockwise, then  $L^{-1}$  is a rotation by 315 degrees counterclockwise.
- e) The zero map ( $0\mathbf{V} = 0$  for all vectors  $\mathbf{V}$ ) is invertible.
- f) The identity map ( $I\mathbf{V} = \mathbf{V}$  for all vectors  $\mathbf{V}$ ) is invertible.
- g) If  $L$  is invertible, then  $L^{-1}0 = 0$ .
- h) If  $L\mathbf{V} = 0$  for some non-zero vector  $\mathbf{V}$ , then  $L$  is not invertible.
- i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse:  $L = L^{-1}$ .

26. Let  $L$ ,  $M$ , and  $N$  be linear maps from the (two dimensional) plane to the plane given in terms of the standard  $\mathbf{i}$ ,  $\mathbf{j}$  basis vectors by:

$$L\mathbf{i} = \mathbf{j}, \quad L\mathbf{j} = -\mathbf{i} \quad (\text{rotation by 90 degrees counterclockwise})$$

$$M\mathbf{i} = -\mathbf{i}, \quad M\mathbf{j} = \mathbf{j} \quad (\text{reflection across the vertical axis})$$

$$N\mathbf{v} = -\mathbf{v} \quad (\text{reflection across the origin})$$

- a) Draw pictures describing the actions of the maps  $L$ ,  $M$ , and  $N$  and the compositions:  $LM$ ,  $ML$ ,  $LN$ ,  $NL$ ,  $MN$ , and  $NM$ .
- b) Which pairs of these maps commute?
- c) Which of the following identities are correct—and why?
  - 1)  $L^2 = N$     2)  $N^2 = I$     3)  $L^4 = I$     4)  $L^5 = L$
  - 5)  $M^2 = I$     6)  $M^3 = M$     7)  $MNM = N$     8)  $NMN = L$
- d) Find matrices representing each of the linear maps  $L$ ,  $M$ , and  $N$ .

27. Give a proof or counterexample the following. In each case your answers should be brief.

- a) Suppose that  $u$ ,  $v$  and  $w$  are vectors in a vector space  $V$  and  $T : V \rightarrow W$  is a linear map. If  $u$ ,  $v$  and  $w$  are linearly dependent, is it true that  $T(u)$ ,  $T(v)$  and  $T(w)$  are linearly dependent? Why?
- b) If  $T : \mathbb{R}^6 \rightarrow \mathbb{R}^4$  is a linear map, is it possible that the nullspace of  $T$  is one dimensional?

28. Let  $V$  be a vector space and  $\ell : V \rightarrow \mathbb{R}$  be a linear map. If  $z \in V$  is not in the nullspace of  $\ell$ , show that every  $x \in V$  can be decomposed uniquely as  $x = v + cz$ , where  $v$  is in the nullspace of  $\ell$  and  $c$  is a scalar. [MORAL: The nullspace of a linear functional has codimension one.]

29. Let  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  and  $B : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , so  $BA : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . Show that  $BA$  can *not* be invertible.

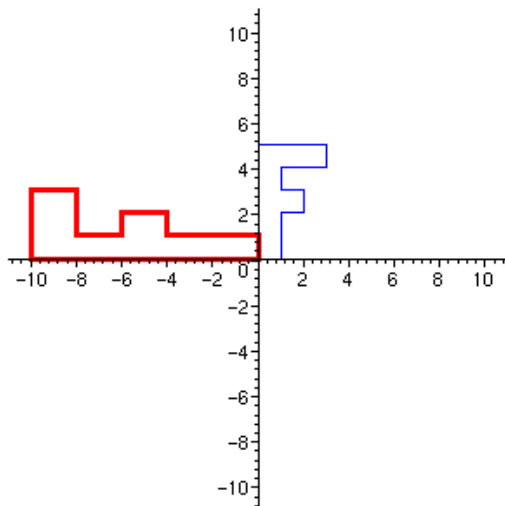
30. Let  $\mathcal{M}_{(3,2)}$  be the linear space of all  $3 \times 2$  real matrices and let the linear map  $L : \mathcal{M}_{(3,2)} \rightarrow \mathbb{R}^5$  be *onto*. Compute the dimension of the nullspace of  $L$ .

31. Think of the matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  as mapping one plane to another.

- a) If two lines in the first plane are parallel, show that after being mapped by  $A$  they are also parallel – although they might coincide.

b) Let  $Q$  be the unit square:  $0 < x < 1, 0 < y < 1$  and let  $Q'$  be its image under this map  $A$ . Show that the  $\text{area}(Q') = |ad - bc|$ . [More generally, the area of any region is magnified by  $|ad - bc|$ , which is called the *determinant* of  $A$ .]

32. a). Find a linear map of the plane,  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that does the following transformation of the letter **F** (here the smaller **F** is transformed to the larger one):



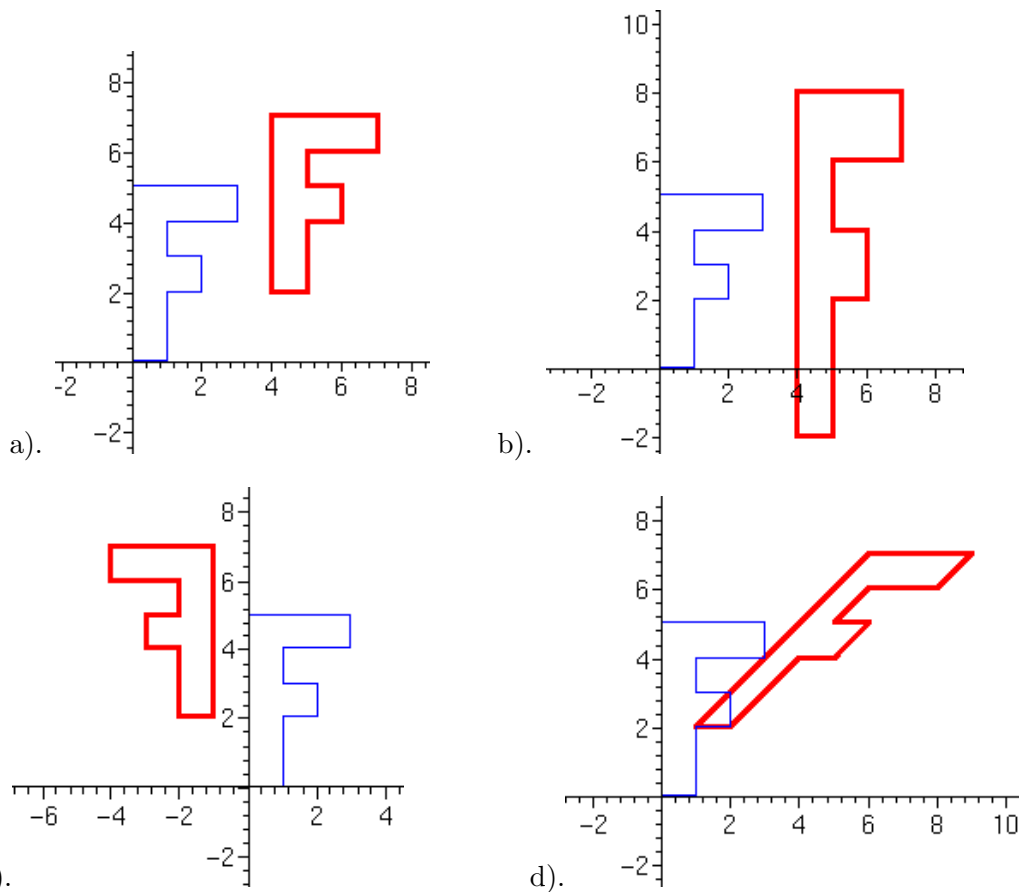
b). Find a linear map of the plane that inverts this map, that is, it maps the larger **F** to the smaller.

33. Linear maps  $F(X) = AX$ , where  $A$  is a matrix, have the property that  $F(0) = A0 = 0$ , so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

$$F(X) = V + AX,$$

where  $V$  is a vector. Note that  $F(0) = V$ .

Find the vector  $V$  and the matrix  $A$  that describe each of the following mappings [here the light blue  $F$  is mapped to the dark red  $F$ ].



34. Find all linear maps  $L: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  whose kernel is exactly the plane  $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0\}$ .

35. Let  $A$  be a matrix, not necessarily square. Say  $\mathbf{V}$  and  $\mathbf{W}$  are particular solutions of the equations  $A\mathbf{V} = \mathbf{Y}_1$  and  $A\mathbf{W} = \mathbf{Y}_2$ , respectively, while  $\mathbf{Z} \neq 0$  is a solution of the homogeneous equation  $A\mathbf{Z} = 0$ . Answer the following in terms of  $\mathbf{V}$ ,  $\mathbf{W}$ , and  $\mathbf{Z}$ .

- Find some solution of  $A\mathbf{X} = 3\mathbf{Y}_1$ .
- Find some solution of  $A\mathbf{X} = -5\mathbf{Y}_2$ .
- Find some solution of  $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$ .
- Find another solution (other than  $\mathbf{Z}$  and  $0$ ) of the homogeneous equation  $A\mathbf{X} = 0$ .
- Find *two* solutions of  $A\mathbf{X} = \mathbf{Y}_1$ .
- Find another solution of  $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$ .
- If  $A$  is a square matrix, then  $\det A = ?$
- If  $A$  is a square matrix, for any given vector  $\mathbf{W}$  can one always find at least one solution of  $A\mathbf{X} = \mathbf{W}$ ? Why?

36. Let  $V$  be an  $n$ -dimensional vector space and  $T : V \rightarrow V$  a linear transformation such that the image and kernel of  $T$  are identical.
- Prove that  $n$  is even.
  - Give an example of such a linear transformation  $T$ .
37. Let  $V \subset \mathbb{R}^{11}$  be a linear subspace of dimension 4 and consider the family  $\mathcal{A}$  of all linear maps  $L : \mathbb{R}^{11} \rightarrow \mathbb{R}^9$  each of whose nullspace contain  $V$ .  
Show that  $\mathcal{A}$  is a linear space and compute its dimension.
38. Let  $L$  be a  $2 \times 2$  matrix. For each of the following give a proof or counterexample.
- If  $L^2 = 0$  then  $L = 0$ .
  - If  $L^2 = L$  then either  $L = 0$  or  $L = I$ .
  - If  $L^2 = I$  then either  $L = I$  or  $L = -I$ .
39. Find an example of  $2 \times 2$  matrices  $A$  and  $B$  so that  $AB = 0$  but  $BA \neq 0$ .
40. Let  $A$  and  $B$  be  $n \times n$  matrices with the property that  $AB = 0$ . For each of the following give a proof or counterexample.
- Every eigenvector of  $B$  is also an eigenvector of  $A$ .
  - At least one eigenvector of  $B$  is also an eigenvector of  $A$ .
41. Say  $A \in M(n, \mathbb{F})$  has rank  $k$ . Define
- $$\mathcal{L} := \{ B \in M(n, \mathbb{F}) \mid BA = 0 \} \quad \text{and} \quad \mathcal{R} := \{ C \in M(n, \mathbb{F}) \mid AC = 0 \}.$$
- Show that  $\mathcal{L}$  and  $\mathcal{R}$  are linear spaces and compute their dimensions.
42. Let  $A$  and  $B$  be  $n \times n$  matrices.
- Show that the rank  $(AB) \leq \text{rank}(A)$ . Give an example where strict inequality can occur.
  - Show that  $\dim(\ker AB) \geq \dim(\ker A)$ . Give an example where strict inequality can occur.
43. Let  $V$  be the linear space of smooth real-valued functions and  $L : V \rightarrow V$  the linear map defined by  $Lu := u'' + u$ .
- Compute  $L(e^{2x})$  and  $L(x)$ .

b) Find particular solutions of the inhomogeneous equations

$$a). u'' + u = 7e^{2x}, \quad b). w'' + w = 4x, \quad c). z'' + z = 7e^{2x} - 3x$$

c) Find the kernel (=nullspace) of  $L$ . What is its dimension?

44. Let  $\mathcal{P}_N$  be the linear space of polynomials of degree at most  $N$  and  $L : \mathcal{P}_N \rightarrow \mathcal{P}_N$  the linear map defined by  $Lu := au'' + bu' + cu$ , where  $a$ ,  $b$ , and  $c$  are constants. Assume  $c \neq 0$ .

a) Compute  $L(x^k)$ .

b) Show that nullspace (=kernel) of  $L : \mathcal{P}_N \rightarrow \mathcal{P}_N$  is 0.

c) Show that for every polynomial  $q(x) \in \mathcal{P}_N$  there is one and only one solution  $p(x) \in \mathcal{P}_N$  of the ODE  $Lp = q$ .

d) Find some solution  $v(x)$  of  $v'' + v = x^2 - 1$ .

45. Let  $\mathcal{P}_1$  be the linear space of real polynomials of degree at most one, so a typical element is  $p(x) := a + bx$ , where  $a$  and  $b$  are real numbers. The derivative,  $D : \mathcal{P}_1 \rightarrow \mathcal{P}_1$  is, as you should expect, the map  $DP(x) = b = b + 0x$ . Using the basis  $e_1(x) := 1$ ,  $e_2(x) := x$  for  $\mathcal{P}_1$ , we have  $p(x) = ae_1(x) + be_2(x)$  so  $Dp = be_1$ .

Using this basis, find the  $2 \times 2$  matrix  $M$  for  $D$ . Note the obvious property  $D^2p = 0$  for any polynomial  $p$  of degree at most 1. Does  $M$  also satisfy  $M^2 = 0$ ? Why should you have expected this?

46. Let  $\mathcal{P}_2$  be the space of polynomials of degree at most 2.

a) Find a basis for this space.

b) Let  $D : \mathcal{P}_2 \rightarrow \mathcal{P}_2$  be the derivative operator  $D = d/dx$ . Using the basis you picked in the previous part, write  $D$  as a matrix. Compute  $D^3$  in this situation. Why should you have predicted this without computation?

47. a) Let  $\{e_1, e_2, \dots, e_n\}$  be the standard basis in  $\mathbb{R}^n$  and let  $\{v_1, v_2, \dots, v_n\}$  be another basis in  $\mathbb{R}^n$ . Find a matrix  $A$  that maps the standard basis to this other basis.

b) Let  $\{w_1, w_2, \dots, w_n\}$  be yet another basis for  $\mathbb{R}^n$ . Find a matrix that maps the  $\{v_j\}$  basis to the  $\{w_j\}$  basis. Write this matrix explicitly if both bases are orthonormal.

48. Let  $\mathcal{S} \subset \mathbb{R}^3$  be the subspace spanned by the two vectors  $v_1 = (1, -1, 0)$  and  $v_2 = (1, -1, 1)$  and let  $\mathcal{T}$  be the *orthogonal complement* of  $\mathcal{S}$  (so  $\mathcal{T}$  consists of all the vectors orthogonal to  $\mathcal{S}$ ).

a) Find an orthogonal basis for  $\mathcal{S}$  and use it to find the  $3 \times 3$  matrix  $P$  that projects vectors orthogonally into  $\mathcal{S}$ .

- b) Find an orthogonal basis for  $\mathcal{T}$  and use it to find the  $3 \times 3$  matrix  $Q$  that projects vectors orthogonally into  $\mathcal{T}$ .
- c) Verify that  $P = I - Q$ . How could you have seen this in advance?
49. a) Let  $\mathbf{v} := (\alpha, \beta, \gamma)$  and  $\mathbf{x} := (x, y, z)$  be any vectors in  $\mathbb{R}^3$ . Viewed as column vectors, find a  $3 \times 3$  matrix  $A_{\mathbf{v}}$  so that the *cross product*  $\mathbf{v} \times \mathbf{x} = A_{\mathbf{v}}\mathbf{x}$ .

ANSWER:

$$\mathbf{v} \times \mathbf{x} = A_{\mathbf{v}}\mathbf{x} = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where the anti-symmetric matrix  $A_{\mathbf{v}}$  is defined by the above formula.

- b) From this, one has  $\mathbf{v} \times (\mathbf{v} \times \mathbf{x}) = A_{\mathbf{v}}(\mathbf{v} \times \mathbf{x}) = A_{\mathbf{v}}^2\mathbf{x}$  (why?). Combined with the cross product identity  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}$ , show that

$$A_{\mathbf{v}}^2\mathbf{x} = \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{v} - \|\mathbf{v}\|^2 \mathbf{x}.$$

- c) If  $\mathbf{n} = (a, b, c)$  is a unit vector, use this formula to show that (perhaps surprisingly) the orthogonal projection of  $\mathbf{x}$  into the plane perpendicular to  $\mathbf{n}$  is given by

$$\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n} = -A_{\mathbf{n}}^2\mathbf{x} = - \begin{pmatrix} -b^2 - c^2 & ab & ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{pmatrix}$$

(See also Problem 123, 151, 150).

50. Let  $V$  be a vector space with  $\dim V = 10$  and let  $L : V \rightarrow V$  be a linear transformation. Consider  $L^k : V \rightarrow V$ ,  $k = 1, 2, 3, \dots$ . Let  $r_k = \dim(\text{Im } L^k)$ , that is,  $r_k$  is the dimension of the image of  $L^k$ ,  $k = 1, 2, \dots$

Give an example of a linear transformation  $L : V \rightarrow V$  (or show that there is no such transformation) for which:

- a)  $(r_1, r_2, \dots) = (10, 9, \dots)$ ;
- b)  $(r_1, r_2, \dots) = (8, 5, \dots)$ ;
- c)  $(r_1, r_2, \dots) = (8, 6, 4, 4, \dots)$ .
51. Let  $\mathcal{S}$  be the linear space of infinite sequences of real numbers  $x := (x_1, x_2, \dots)$ . Define the linear map  $L : \mathcal{S} \rightarrow \mathcal{S}$  by

$$Lx := (x_1 + x_2, x_2 + x_3, x_3 + x_4, \dots).$$

- a) Find a basis for the nullspace of  $L$ . What is its dimension?
- b) What is the image of  $L$ ? Justify your assertion.
- c) Compute the eigenvalues of  $L$  and an eigenvector corresponding to each eigenvalue.

### Rank One Matrices

52. Let  $A = (a_{ij})$  be an  $n \times n$  matrix whose rank is 1 and let  $v := (v_1, \dots, v_n)$  be a basis for the image of  $A$ .
- a) Show that  $a_{ij} = v_i w_j$  for some vector  $w := (w_1, \dots, w_n)$ .
  - b) If  $A$  has a non-zero eigenvalue  $\lambda_1$ , show that
  - c) If the vector  $z = (z_1, \dots, z_n)$  satisfies  $\langle z, w \rangle = 0$ , show that  $z$  is an eigenvector with eigenvalue  $\lambda = 0$ .
  - d) If  $\text{trace}(A) \neq 0$ , show that  $\lambda = \text{trace}(A)$  is an eigenvalue of  $A$ . What is the corresponding eigenvector?
  - e) If  $\text{trace}(A) \neq 0$ , prove that  $A$  is similar to the  $n \times n$  matrix

$$\begin{pmatrix} c & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

where  $c = \text{trace}(A)$

- f) If  $\text{trace}(A) = 1$ , show that  $A$  is a projection, that is,  $A^2 = A$ .
  - g) What can you say if  $\text{trace}(A) = 0$ ?
53. Let  $A$  be the rank one  $n \times n$  matrix  $A = (v_i v_j)$ , where  $V := (v_1, \dots, v_n)$  is a non-zero real vector.
- a) Find its eigenvalues and eigenvectors.
  - b) Find a formula for  $(I + A)^{-1}$ . [ANSWER:  $(I + A)^{-1} = I - \frac{1}{1 + \|V\|^2} A$ .]
  - c) Find the eigenvalues and eigenvectors for  $A + cI$ , where  $c \in \mathbb{R}$ .

### Algebra of Matrices

54. Every real upper triangular  $n \times n$  matrix  $(a_{ij})$  with  $a_{ii} = 1$ ,  $i = 1, 2, \dots, n$  is invertible. Proof or counterexample.
55. Let  $L: V \rightarrow V$  be a linear map on a vector space  $V$ .

- a) Show that  $\ker L \subset \ker L^2$  and, more generally,  $\ker L^k \subset \ker L^{k+1}$  for all  $k \geq 1$ .
- b) If  $\ker L^j = \ker L^{j+1}$  for some integer  $j$ , show that  $\ker L^k = \ker L^{k+1}$  for all  $k \geq j$ . Does your proof require that  $V$  is finite dimensional?
- c) Let  $A$  be an  $n \times n$  matrix. If  $A^j = 0$  for some integer  $j$ , show that  $A^n = 0$ .
56. Let  $L: V \rightarrow V$  be a linear map on a vector space  $V$  and  $z \in V$  a vector with the property that  $L^{k-1}z \neq 0$  but  $L^kz = 0$ . Show that  $z, Lz, \dots, L^{k-1}z$  are linearly independent.
57. Let  $A, B$ , and  $C$  be any  $n \times n$  matrices.
- a) Show that  $\text{trace}(AB) = \text{trace}(BA)$ .
- b) Show that  $\text{trace}(ABC) = \text{trace}(CAB) = \text{trace}(BCA)$ .
- c)  $\text{trace}(ABC) \stackrel{?}{=} \text{trace}(BAC)$ . Proof or counterexample.
58. Let  $A$  be an  $n \times n$  matrix. If  $AB = BA$  for *all* invertible matrices  $B$ , show that  $A = cI$  for some scalar  $c$ .
59. There are no square matrices  $A, B$  with the property that  $AB - BA = I$ . Proof or counterexample.
60. Let  $A: \mathbb{R}^\ell \rightarrow \mathbb{R}^n$  and  $B: \mathbb{R}^k \rightarrow \mathbb{R}^\ell$ . Prove the following.
- a)  $\text{rank } A + \text{rank } B - \ell \leq \text{rank } AB \leq \min\{\text{rank } A, \text{rank } B\}$ .
- b)  $|\text{rank } A - \text{rank } B| \leq \text{rank } AB$ . [HINT: Observe that  $\text{rank}(AB) = \text{rank } A|_{\text{image}(B)}$ .]

### Eigenvalues and Eigenvectors

61. Let  $A$  be an invertible matrix. If  $\mathbf{V}$  is an eigenvector of  $A$ , show it is also an eigenvector of both  $A^2$  and  $A^{-2}$ . What are the corresponding eigenvalues?
62. Let  $C$  be a  $2 \times 2$  matrix of real numbers. Give a proof or counterexample to each of the following assertions:
- a)  $\det(C^2)$  is non-negative.
- b)  $\text{trace}(C^2)$  is non-negative.
- c) All of the elements of  $C^2$  are non-negative.
- d) All the eigenvalues of  $C^2$  are non-negative.
- e) If  $C$  has two distinct eigenvalues, then so does  $C^2$ .

63. Let  $A \in M(n, \mathbb{F})$  have an eigenvalue  $\lambda$  with corresponding eigenvector  $v$ .

True or False

- a)  $-v$  is an eigenvector of  $-A$  with eigenvalue  $-\lambda$ .
- b) If  $v$  is also an eigenvector of  $B \in M(n, \mathbb{F})$  with eigenvalue  $\mu$ , then  $\lambda\mu$  is an eigenvalue of  $AB$ .
- c) Let  $c \in \mathbb{F}$ . Then  $(\lambda + c)^2$  is an eigenvalue of  $A^2 + 2cA + c^2I$ .
- d) Let  $\mu$  be an eigenvalue of  $B \in M(n, \mathbb{F})$ , Then  $\lambda + \mu$  is an eigenvalue of  $A + B$ .
- e) Let  $c \in \mathbb{F}$ . Then  $c\lambda$  is an eigenvalue of  $cA$ .

64. Let  $M$  be a  $2 \times 2$  matrix with the property that the sum of each of the rows and also the sum of each of the columns is the *same* constant  $c$ . Which (if any) any of the vectors

$$U := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad V := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad W := \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

must be an eigenvector of  $M$ ?

65. Let  $A$  and  $B$  be  $n \times n$  complex matrices that commute:  $AB = BA$ . If  $\lambda$  is an eigenvalue of  $A$ , let  $\mathcal{V}_\lambda$  be the subspace of all eigenvectors having this eigenvalue.

- a) Show there is a vector  $v \in \mathcal{V}_\lambda$  that is also an eigenvector of  $B$ , possibly with a different eigenvalue.
- b) Give an example showing that some vectors in  $\mathcal{V}_\lambda$  may not be eigenvectors of  $B$ .
- c) If all the eigenvalues of  $A$  are distinct (so each has algebraic multiplicity one), show that there is a basis in which both  $A$  and  $B$  are diagonal. Also, give an example showing this may be false if some eigenvalue of  $A$  has multiplicity greater than one.

66. Let  $A$  be a  $3 \times 3$  matrix with eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  and corresponding linearly independent eigenvectors  $V_1, V_2, V_3$  which we can therefore use as a basis.

- a) If  $X = aV_1 + bV_2 + cV_3$ , compute  $AX, A^2X$ , and  $A^{35}X$  in terms of  $\lambda_1, \lambda_2, \lambda_3, V_1, V_2, V_3, a, b$  and  $c$  (only).
- b) If  $\lambda_1 = 1, |\lambda_2| < 1$ , and  $|\lambda_3| < 1$ , compute  $\lim_{k \rightarrow \infty} A^k X$ . Explain your reasoning clearly.

67. Let  $Z$  be a complex square matrix whose self-adjoint part is positive definite, so  $Z + Z^*$  is positive definite.

- a) Show that the eigenvalues of  $Z$  have positive real part.
- b) Is the converse true? Proof or counterexample.

68. Let  $A$  be an  $n \times k$  matrix and  $B$  a  $k \times n$  matrix. Then both  $AB$  and  $BA$  are square matrices.
- If  $\lambda \neq 0$  is an eigenvalue of  $AB$ , show it is also an eigenvalue of  $BA$ . In particular, the non-zero eigenvalues of  $A^*A$  and  $AA^*$  agree.
  - If  $v_1, \dots, v_k$  are linearly independent eigenvectors of  $BA$  corresponding to the same eigenvalue,  $\lambda \neq 0$ , show that  $Av_1, \dots, Av_k$  are linearly independent eigenvectors of  $AB$  corresponding to  $\lambda$ . Thus the eigenspaces of  $AB$  and  $BA$  corresponding to a non-zero eigenvalue have the same geometric multiplicity.
  - (This gives a sharper result to the first part) We seek a formula relating the characteristic polynomials  $p_{AB}(\lambda)$  of  $AB$  and  $p_{BA}(\lambda)$  of  $BA$ , respectively. Show that

$$\lambda^k p_{AB}(\lambda) = \lambda^n p_{BA}(\lambda).$$

In particular if  $A$  and  $B$  are square, then  $AB$  and  $BA$  have the same characteristic polynomial. [SUGGESTION: One approach uses *block matrices*: let  $P = \begin{pmatrix} \lambda I_n & A \\ B & I_k \end{pmatrix}$  and  $Q = \begin{pmatrix} I_n & 0 \\ -B & \lambda I_k \end{pmatrix}$ , where  $I_m$  is the  $m \times m$  identity matrix. Then use  $\det(PQ) = \det(QP)$ .]

69. Let  $A$  be a square matrix with the property that the sum of the elements in each of its columns is 1. Show that  $\lambda = 1$  is an eigenvalue of  $A$ . [These matrices arise in the study of Markov chains.]
70. Given any real monic polynomial  $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ , find an  $n \times n$  real matrix with this as its characteristic polynomial. [This is related to writing an  $n^{\text{th}}$  order linear ordinary differential equation as a system of first order linear equations.]
71. Compute the value of the determinant of the  $3 \times 3$  complex matrix  $X$ , provided that  $\text{tr}(X) = 1$ ,  $\text{tr}(X^2) = -3$ ,  $\text{tr}(X^3) = 4$ . [Here  $\text{tr}(A)$  denotes the the *trace*, that is, the sum of the diagonal entries of the matrix  $A$ .]

72. Let  $A := \begin{pmatrix} 4 & 4 & 4 \\ -2 & -3 & -6 \\ 1 & 3 & 6 \end{pmatrix}$ . Compute

- the characteristic polynomial,
- the eigenvalues,
- one* of the corresponding eigenvectors.

73. Let  $A \in M(4, \mathbb{F})$ , where here  $\mathbb{F}$  is any field. Let  $\chi_A$  be the characteristic polynomial of  $A$  and  $p(t) := t^4 + 1 \in \mathbb{F}[t]$ .

True or False?

- a) If  $\chi_A = p$ , then  $A$  is invertible.
- b) If  $\chi_A = p$ , then  $A$  is diagonalizable over  $\mathbb{F}$ .
- c) If  $p(B) = 0$  for some matrix  $B \in M(8, \mathbb{F})$ , then  $P$  is the characteristic polynomial of  $B$ .
- d) There is a unique monic polynomial  $q \in \mathbb{F}[t]$  of degree 4 such that  $q(A) = 0$ .
- e) A matrix  $B \in M(n, \mathbb{F})$  is always nilpotent if its minimal polynomial is  $t^k$  for some integer  $k \geq 1$ .

74. Let  $A$  be a square matrix. In the following, a sequence of matrices  $C_j$  converges if all of its elements converge.

Prove that the following are equivalent:

- (i)  $A^k \rightarrow 0$  as  $k \rightarrow \infty$  [each of the elements of  $A^k$  converge to zero].
- (ii) All the eigenvalues  $\lambda_j$  of  $A$  have  $|\lambda_j| < 1$ .
- (iii) The matrix geometric series  $\sum_0^\infty A^k$  converges to  $(I - A)^{-1}$ .

75. Let  $A$  be a square matrix and let  $\|B\|$  be any norm on matrices [one example is  $\|B\| = \max_{i,j} |b_{ij}|$ ]. To what extent are the conditions in the previous problem also equivalent to the condition that  $\|A^k\| \rightarrow 0$ ?

- 76. a) Prove that the set of invertible real  $2 \times 2$  matrices is dense in the set of all real  $2 \times 2$  matrices.
- b) The set of diagonalizable  $2 \times 2$  matrices dense in the set of all real  $2 \times 2$  matrices. Proof or counterexample?

77. [SPECTRAL MAPPING THEOREM] Let  $A$  be a square matrix.

- a) If  $A(A - I)(A - 2I) = 0$ , show that the only possible eigenvalues of  $A$  are  $\lambda = 0$ ,  $\lambda = 1$ , and  $\lambda = 2$ .
- b) Let  $p$  any polynomial. Show that the eigenvalues of the matrix  $p(A)$  are precisely the numbers  $p(\lambda_j)$ , where the  $\lambda_j$  are the eigenvalues of  $A$ .

## Inner Products and Quadratic Forms

78. Let  $V, W$  be vectors in the plane  $\mathbb{R}^2$  with lengths  $\|V\| = 3$  and  $\|W\| = 5$ . What are the maxima and minima of  $\|V + W\|$ ? When do these occur?

79. Let  $V, W$  be vectors in  $\mathbb{R}^n$ .
- Show that the Pythagorean relation  $\|V + W\|^2 = \|V\|^2 + \|W\|^2$  holds if and only if  $V$  and  $W$  are orthogonal.
  - Prove the *parallelogram identity*  $\|V + W\|^2 + \|V - W\|^2 = 2\|V\|^2 + 2\|W\|^2$  and interpret it geometrically.
80. Let  $A = (-6, 3)$ ,  $B = (2, 7)$ , and  $C$  be the vertices of a triangle. Say the altitudes through the vertices  $A$  and  $B$  intersect at  $Q = (2, -1)$ . Find the coordinates of  $C$ .
- [The *altitude* through a vertex of a triangle is a straight line through the vertex that is perpendicular to the opposite side — or an extension of the opposite side. Although not needed here, the three altitudes always intersect in a single point, sometimes called the orthocenter of the triangle.]
81. Find all vectors in the plane (through the origin) spanned by  $\mathbf{V} = (1, 1 - 2)$  and  $\mathbf{W} = (-1, 1, 1)$  that are perpendicular to the vector  $\mathbf{Z} = (2, 1, 2)$ .
82. In  $\mathbb{R}^3$ , let  $V$  be a non-zero vector and  $X_0$  and  $P$  points.
- Find the equation of the plane through  $X_0$  that is orthogonal to  $V$ , so  $V$  is a normal vector to this plane.
  - Compute the distance from this plane to the origin and also to the point  $P$ .
  - Let  $S$  be the sphere centered at  $P$  with radius  $r$ . For which value(s) of  $r$  is this sphere tangent to the above plane?
83. Let  $U, V, W$  be orthogonal vectors and let  $Z = aU + bV + cW$ , where  $a, b, c$  are scalars.
- (Pythagoras) Show that  $\|Z\|^2 = a^2\|U\|^2 + b^2\|V\|^2 + c^2\|W\|^2$ .
  - Find a formula for the coefficient  $a$  in terms of  $U$  and  $Z$  only. Then find similar formulas for  $b$  and  $c$ . [Suggestion: take the inner product of  $Z = aU + bV + cW$  with  $U$ ].

REMARK The resulting simple formulas are one reason that orthogonal vectors are easier to use than more general vectors. This is vital for Fourier series.

- Solve the following equations:

$$\begin{aligned} x + y + z + w &= 2 \\ x + y - z - w &= 3 \\ x - y + z - w &= 0 \\ x - y - z + w &= -5 \end{aligned}$$

[Suggestion: Observe that the columns vectors in the coefficient matrix are orthogonal.]

84. [LINEAR FUNCTIONALS] In  $\mathbb{R}^n$  with the usual inner product, a *linear functional*  $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$  is just a linear map into the reals (in a complex vector space, it maps into the complex numbers  $\mathbb{C}$ ). Define the norm,  $\|\ell\|$ , as

$$\|\ell\| := \max_{\|x\|=1} |\ell(x)|.$$

- Show that the set of linear functionals with this norm is a normed linear space.
- If  $v \in \mathbb{R}^n$  is a given vector, define  $\ell(x) = \langle x, v \rangle$ . Show that  $\ell$  is a linear functional and that  $\|\ell\| = \|v\|$ .
- [REPRESENTATION OF A LINEAR FUNCTIONAL] Let  $\ell$  be any linear functional. Show there is a unique vector  $v \in \mathbb{R}^n$  so that  $\ell(x) := \langle x, v \rangle$ .
- [EXTENSION OF A LINEAR FUNCTIONAL] Let  $U \subset \mathbb{R}^n$  be a subspace of  $\mathbb{R}^n$  and  $\ell$  a linear functional defined on  $U$  with norm  $\|\ell\|_U$ . Show there is a unique extension of  $\ell$  to  $\mathbb{R}^n$  with the property that  $\|\ell\|_{\mathbb{R}^n} = \|\ell\|_U$ .  
[In other words define  $\ell$  on all of  $\mathbb{R}^n$  so that on  $U$  this extended definition agrees with the original definition and so that its norm is unchanged].

85. a) Let  $A$  be a positive definite  $n \times n$  real matrix,  $b \in \mathbb{R}^n$ , and consider the quadratic polynomial

$$Q(x) := \frac{1}{2} \langle x, Ax \rangle - \langle b, x \rangle.$$

Show that  $Q$  is bounded below, that is, there is a constant  $m$  so that  $Q(x) \geq m$  for all  $x \in \mathbb{R}^n$ .

- Show that  $Q$  blows up at infinity by showing that there are positive constants  $R$  and  $c$  so that if  $\|x\| \geq R$ , then  $Q(x) \geq c\|x\|^2$ .
- If  $x_0 \in \mathbb{R}^n$  minimizes  $Q$ , show that  $Ax_0 = b$ . [Moral: One way to solve  $Ax = b$  is to minimize  $Q$ .]

86. Let  $A$  be a square matrix of real numbers whose columns are (non-zero) orthogonal vectors.

- Show that  $A^T A$  is a diagonal matrix — whose inverse is thus obvious to compute.
- Use this observation (or any other method) to discover a simple general formula for the inverse,  $A^{-1}$  involving only its transpose,  $A^T$ , and  $(A^T A)^{-1}$ . In the special case where the columns of  $A$  are orthonormal, your formula should reduce to  $A^{-1} = A^T$ .
- Apply this to again solve the equations in Problem (83c).

87. [Gram-Schmidt Orthogonalization]

- a) Let  $A := \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . Briefly show that the bilinear map  $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$  defined by  $(x, y) \mapsto x^T A y$  gives a scalar product.
- b) Let  $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$  be the linear functional  $\alpha : (x_1, x_2, x_3) \mapsto x_1 + x_2$  and let  $v_1 := (-1, 1, 1)$ ,  $v_2 := (2, -2, 0)$  and  $v_3 := (1, 0, 0)$  be a basis of  $\mathbb{R}^3$ . Using the scalar product of the previous part, find an orthonormal basis  $\{e_1, e_2, e_3\}$  of  $\mathbb{R}^3$  with  $e_1 \in \text{span}\{v_1\}$  and  $e_2 \in \ker \alpha$ .

88. Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be a linear map defined by the matrix  $A$ . If the matrix  $B$  satisfies the relation  $\langle AX, Y \rangle = \langle X, BY \rangle$  for all vectors  $X \in \mathbb{R}^n$ ,  $Y \in \mathbb{R}^k$ , show that  $B$  is the transpose of  $A$ , so  $B = A^T$ . [This basic property of the transpose,

$$\langle AX, Y \rangle = \langle X, AY \rangle,$$

is the *only* reason the transpose is important.]

89. Proof or counterexample. Here  $v, w, z$  are vectors in a real inner product space  $H$ .
- a) Let  $v, w, z$  be vectors in a real inner product space. If  $\langle v, w \rangle = 0$  and  $\langle v, z \rangle = 0$ , then  $\langle w, z \rangle = 0$ .
- b) If  $\langle v, z \rangle = \langle w, z \rangle$  for all  $z \in H$ , then  $v = w$ .
- c) If  $A$  is an  $n \times n$  symmetric matrix then  $A$  is invertible.

90. In  $\mathbb{R}^4$ , compute the distance from the point  $(1, -2, 0, 3)$  to the hyperplane  $x_1 + 3x_2 - x_3 + x_4 = 3$ .

91. Find the (orthogonal) projection of  $\mathbf{x} := (1, 2, 0)$  into the following subspaces:

- a) The line spanned by  $\mathbf{u} := (1, 1, -1)$ .
- b) The plane spanned by  $\mathbf{u} := (0, 1, 0)$  and  $\mathbf{v} := (0, 0, -2)$
- c) The plane spanned by  $\mathbf{u} := (0, 1, 1)$  and  $\mathbf{v} := (0, 1, -2)$
- d) The plane spanned by  $\mathbf{u} := (1, 0, 1)$  and  $\mathbf{v} := (1, 1, -1)$
- e) The plane spanned by  $\mathbf{u} := (1, 0, 1)$  and  $\mathbf{v} := (2, 1, 0)$ .
- f) The subspace spanned by  $\mathbf{u} := (1, 0, 1)$ ,  $\mathbf{v} := (2, 1, 0)$  and  $\mathbf{w} := (1, 1, 0)$ .

92. Let  $\mathcal{S} \subset \mathbb{R}^4$  be the vectors  $X = (x_1, x_2, x_3, x_4)$  that satisfy  $x_1 + x_2 - x_3 + x_4 = 0$ .

- a) What is the dimension of  $\mathcal{S}$ ?
- b) Find a basis for the orthogonal complement of  $\mathcal{S}$ .

93. Let  $\mathcal{S} \subset \mathbb{R}^4$  be the subspace spanned by the two vectors  $v_1 = (1, -1, 0, 1)$  and  $v_2 = (0, 0, 1, 0)$  and let  $\mathcal{T}$  be the orthogonal complement of  $\mathcal{S}$ .
- Find an orthogonal basis for  $\mathcal{T}$ .
  - Compute the orthogonal projection of  $(1, 1, 1, 1)$  into  $\mathcal{S}$ .
94. Let  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear map with the property that  $L\mathbf{v} \perp \mathbf{v}$  for every  $\mathbf{v} \in \mathbb{R}^3$ . Prove that  $L$  cannot be invertible. Is a similar assertion true for a linear map  $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ?
95. In a complex vector space (with a hermitian inner product), if a matrix  $A$  satisfies  $\langle X, AX \rangle = 0$  for all vectors  $X$ , show that  $A = 0$ . [The previous problem shows that this is false in a real vector space].
96. Using the inner product  $\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$ , for which values of the real constants  $\alpha, \beta, \gamma$  are the quadratic polynomials  $p_1(x) = 1$ ,  $p_2(x) = \alpha + x$ ,  $p_3(x) = \beta + \gamma x + x^2$  orthogonal? [ANSWER:  $p_2(x) = x$ ,  $p_3(x) = x^2 - 1/3$ .]
97. Using the inner product of the previous problem, let  $\mathcal{B} = \{1, x, 3x^2 - 1\}$  be an orthogonal basis for the space  $\mathcal{P}_2$  of quadratic polynomials and let  $\mathcal{S} = \text{span}(x, x^2) \subset \mathcal{P}_2$ . Using the basis  $\mathcal{B}$ , find the linear map  $P : \mathcal{P}_2 \rightarrow \mathcal{P}_2$  that is the orthogonal projection from  $\mathcal{P}_2$  onto  $\mathcal{S}$ .
98. Find an orthogonal basis for the space  $\mathcal{P}_2$  of quadratic polynomials with the inner product
- $$\langle f, g \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1).$$
99. Let  $\mathcal{P}_2$  be the space of polynomials  $p(x) = a + bx + cx^2$  of degree at most 2 with the inner product  $\langle p, q \rangle = \int_{-1}^1 p(x)q(x) dx$ . Let  $\ell$  be the functional  $\ell(p) := p(0)$ . Find  $h \in \mathcal{P}_2$  so that  $\ell(p) = \langle h, p \rangle$  for all  $p \in \mathcal{P}_2$ .
100. Let  $C[-1, 1]$  be the real inner product space consisting of all continuous functions  $f : [-1, 1] \rightarrow \mathbb{R}$ , with the inner product  $\langle f, g \rangle := \int_{-1}^1 f(x)g(x) dx$ . Let  $W$  be the subspace of odd functions, i.e. functions satisfying  $f(-x) = -f(x)$ . Find (with proof) the orthogonal complement of  $W$ .
101. Find the function  $f \in \text{span}\{1, \sin x, \cos x\}$  that minimizes  $\|\sin 2x - f(x)\|$ , where the norm comes from the inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x) dx \quad \text{on } C[-\pi, \pi].$$

102. a) Let  $V \subset \mathbb{R}^n$  be a subspace and  $Z \in \mathbb{R}^n$  a given vector. Find a unit vector  $X$  that is perpendicular to  $V$  with  $\langle X, Z \rangle$  as large as possible.

b) Compute  $\max \int_{-1}^1 x^3 h(x) dx$  where  $h(x)$  is any continuous function on the interval  $-1 \leq x \leq 1$  subject to the restrictions

$$\int_{-1}^1 h(x) dx = \int_{-1}^1 xh(x) dx = \int_{-1}^1 x^2 h(x) dx = 0; \quad \int_{-1}^1 |h(x)|^2 dx = 1.$$

c) Compute  $\min_{a,b,c} \int_{-1}^1 |x^3 - a - bx - cx^2|^2 dx$ .

103. [Dual variational problems] Let  $V \subset \mathbb{R}^n$  be a linear space,  $Q : \mathbb{R}^n \rightarrow V^\perp$  the orthogonal projection into  $V^\perp$ , and  $x \in \mathbb{R}^n$  a given vector. Note that  $Q = I - P$ , where  $P$  is the orthogonal projection into  $V$

a) Show that  $\max_{\{z \perp V, \|z\|=1\}} \langle x, z \rangle = \|Qx\|$ .

b) Show that  $\min_{v \in V} \|x - v\| = \|Qx\|$ .

[Remark: *dual variational problems* are a pair of maximum and minimum problems whose extremal values are equal.]

104. [COMPLETING THE SQUARE] Let

$$\begin{aligned} Q(x) &= \sum a_{ij} x_i x_j + \sum b_i x_i + c \\ &= \langle x, Ax \rangle + \langle b, x \rangle + c \end{aligned}$$

be a real quadratic polynomial so  $x = (x_1, \dots, x_n)$ ,  $b = (b_1, \dots, b_n)$  are real vectors and  $A = a_{ij}$  is a real symmetric  $n \times n$  matrix. Just as in the case  $n = 1$ , if  $A$  is invertible show there is a change of variables  $y = x - v$  (this is a translation by the vector  $v$ ) so that in the new  $y$  variables  $Q$  has the form

$$Q = \langle y, Ay \rangle + \gamma \quad \text{that is,} \quad Q = \sum a_{ij} y_i y_j + \gamma,$$

where  $\gamma$  involves  $A$ ,  $b$ , and  $c$ .

As an example, apply this to  $Q(x) = 2x_1^2 + 2x_1x_2 + 3x_2 - 4$ .

105. Let  $A$  be a positive definite  $n \times n$  real matrix,  $\beta$  a real vector, and  $N$  a real unit vector.

a) For which value(s) of the real scalar  $c$  is the set

$$E := \{ x \in \mathbb{R}^3 \mid \langle x, Ax \rangle + 2\langle \beta, x \rangle + c = 0 \}$$

(an ellipsoid) non-empty? [ANSWER:  $\langle \beta, A^{-1}\beta \rangle \geq c$ . If  $n = 1$ , this of course reduces to a familiar condition.]

- b) For what value(s) of the scalar  $d$  is the plane  $P := \{x \in \mathbb{R}^3 \mid \langle N, x \rangle = d\}$  tangent to the above ellipsoid  $E$  (assumed non-empty)?

[SUGGESTION: First discuss the case where  $A = I$  and  $\beta = 0$ . Then show how by a change of variables, the general case can be reduced to this special case. See also Problem 82].

[ANSWER:

$$d = -\langle N, A^{-1}\beta \rangle \pm \sqrt{\langle N, A^{-1}N \rangle} \sqrt{\langle \beta, A^{-1}\beta \rangle - c}.$$

For  $n = 1$  this is just the solution  $d = \frac{-\beta \pm \sqrt{\beta^2 - ac}}{a}$  of the quadratic equation  $ax^2 + 2\beta x + c = 0$ .]

106. a) Compute

$$\iint_{\mathbb{R}^2} \frac{dx dy}{(1 + 4x^2 + 9y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx dy}{(1 + x^2 + 2xy + 5y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx dy}{(1 + 5x^2 - 4xy + 5y^2)^2}.$$

- b) Compute  $\iint_{\mathbb{R}^2} \frac{dx_1 dx_2}{[1 + \langle x, Cx \rangle]^2}$ , where  $C$  is a positive definite (symmetric)  $2 \times 2$  matrix, and  $x = (x_1, x_2) \in \mathbb{R}^2$ .

- c) Let  $h(t)$  be a given function and say you know that  $\int_0^\infty h(t) dt = \alpha$ . If  $C$  be a positive definite  $2 \times 2$  matrix. Show that

$$\iint_{\mathbb{R}^2} h(\langle x, Cx \rangle) dA = \frac{\pi\alpha}{\sqrt{\det C}}.$$

- d) Compute  $\iint_{\mathbb{R}^2} e^{-(5x^2 - 4xy + 5y^2)} dx dy$ .

- e) Compute  $\iint_{\mathbb{R}^2} e^{-(5x^2 - 4xy + 5y^2 - 2x + 3)} dx dy$ .

- f) Generalize part c) to  $\mathbb{R}^n$  to obtain a formula for

$$\iint_{\mathbb{R}^n} h(\langle x, Cx \rangle) dV,$$

where now  $C$  be a positive definite  $n \times n$  matrix. The answer will involve some integral involving  $h$  and also the “area” of the unit sphere  $S^{n-1} \hookrightarrow \mathbb{R}^n$ .

107. Let  $v_1 \dots v_k$  be vectors in a linear space with an inner product  $\langle \cdot, \cdot \rangle$ . Define the *Gram determinant* by  $G(v_1, \dots, v_k) = \det(\langle v_i, v_j \rangle)$ .

- a) If the  $v_1 \dots v_k$  are orthogonal, compute their Gram determinant.  
 b) Show that the  $v_1 \dots v_k$  are linearly independent if and only if their Gram determinant is not zero.

- c) Better yet, if the  $v_1 \dots v_k$  are linearly independent, show that the symmetric matrix  $(\langle v_i, v_j \rangle)$  is positive definite. In particular, the inequality  $G(v_1, v_2) \geq 0$  is the *Schwarz inequality*.
- d) Conversely, if  $A$  is any  $n \times n$  positive definite matrix, show that there are vectors  $v_1, \dots, v_n$  so that  $A = (\langle v_i, v_j \rangle)$ .
- e) Let  $\mathcal{S}$  denote the subspace spanned by the linearly independent vectors  $w_1 \dots w_k$ . If  $X$  is any vector, let  $P_{\mathcal{S}}X$  be the orthogonal projection of  $X$  into  $\mathcal{S}$ , prove that the distance  $\|X - P_{\mathcal{S}}X\|$  from  $X$  to  $\mathcal{S}$  is given by the formula

$$\|X - P_{\mathcal{S}}X\|^2 = \frac{G(X, w_1, \dots, w_k)}{G(w_1, \dots, w_k)}.$$

108. (continuation) Consider the space of continuous real functions on  $[0, 1]$  with the inner product,  $\langle f, g \rangle := \int_0^1 f(x)g(x) dx$  and related norm  $\|f\|^2 = \langle f, f \rangle$ . Let  $\mathcal{S}_k := \text{span}\{x^{n_1}, x^{n_2}, \dots, x^{n_k}\}$ , where  $\{n_1, n_2, \dots, n_k\}$  are distinct positive integers. Let  $h(x) := x^\ell$  where  $\ell > 0$  is a positive integer – but *not* one of the  $n_j$ 's. Prove that

$$\lim_{k \rightarrow \infty} \|h - P_{\mathcal{S}_k}h\| = 0 \quad \text{if and only if} \quad \sum \frac{1}{n_j} \text{ diverges.}$$

This, combined with the Weierstrass Approximation theorem, proves **Muntz's Theorem**: *Linear combinations of  $x^{n_1}, x^{n_2}, \dots, x^{n_k}$  are dense in  $L_2(0, 1)$  if and only if  $\sum \frac{1}{n_j}$  diverges.*

109. Let  $L : V \rightarrow W$  be a linear map between the linear spaces  $V$  and  $W$ , both having inner products.
- a) Show that  $(\text{im}L)^\perp = \ker L^*$ , where  $L^*$  is the adjoint of  $L$ .
- b) Show that  $\dim \text{im}L = \dim \text{im}L^*$ . [Don't use determinants.]

110. Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be a linear map. Show that

$$\dim \ker(L) - \dim \ker(L^*) = n - k.$$

( $\ker(L^*)$  is often called the *cokernel* of  $L$ ).

111. Let  $U, V$ , and  $W$  be finite dimensional vector spaces with inner products. If  $A : U \rightarrow V$  and  $B : V \rightarrow W$  are linear maps with adjoints  $A^*$  and  $B^*$ , define the linear map  $C : V \rightarrow V$  by

$$C = AA^* + B^*B.$$

If  $U \xrightarrow{A} V \xrightarrow{B} W$  is *exact* [that is,  $\text{image}(A) = \ker(B)$ ], show that  $C : V \rightarrow V$  is invertible.

112. [Bilinear and Quadratic Forms] Let  $\phi$  be a bilinear form over the finite dimensional real vector space  $V$ .  $\phi$  is called *non-degenerate* if  $\phi(x, y) = 0$  for all  $y \in V$  implies  $x = 0$ .

True or False

- If  $\phi$  is non-degenerate, then  $\psi(x, y) := \frac{1}{2}[\phi(x, y) + \phi(y, x)]$  is a scalar product.
- If  $\phi(x, y) = -\phi(y, x)$  for all  $x, y \in V$ , then  $\phi(z, z) = 0$  for all  $z \in V$ .
- If  $\phi$  is symmetric and  $\phi(x, x) = 0$  for all  $x \in V$ , then  $\phi = 0$ .
- Assume the bilinear forms  $\phi$  and  $\psi$  are both symmetric and positive definite. Then  $\{z \in V \mid \phi(x, z)^3 + \psi(y, z)^3 = 0\}$  is a subspace of  $V$ .
- If  $\phi$  and  $\psi$  are bilinear forms over  $V$ , then  $\{z \in V \mid \phi(x, z)^2 + \psi(y, z)^2 = 0\}$  is a subspace of  $V$ .

## Norms and Metrics

113. Let  $\mathcal{P}_n$  be the space of real polynomials with degree at most  $n$ . Write  $p(t) = \sum_{j=0}^n a_j t^j$  and  $q(t) = \sum_{j=0}^n b_j t^j$ .

True or False

- Define  $d: \mathcal{P}_n \times \mathcal{P}_n \rightarrow \mathbb{R}$  by  $d(p, q) := \sum_{j=0}^n |a_j - b_j|$ . Then  $\|p\| = d(p, 0)$  is a norm on  $\mathcal{P}_n$ .
- For  $p \in \mathcal{P}_n$  let  $\|p\| := 0$  when  $p = 0$  and  $\|p\| := \max(0, NP(p))$  for  $p \neq 0$ . Here  $NP(p)$  is the set of all the real zeroes of  $p$ . Claim:  $\|p\|$  is a norm on  $\mathcal{P}_n$ .
- Define a norm  $\|\cdot\|$  on  $\mathcal{P}_n$  by  $\|p\| := \max_{t \in [0, 1]} |p(t)|$ . Then there is a bilinear form  $\phi$  on  $\mathcal{P}_n$  with  $\phi(p, p) = \|p\|^2$  for all  $p \in \mathcal{P}_n$ .
- Let  $\langle \cdot, \cdot \rangle$  be a scalar product on  $\mathcal{P}_n$  and  $\|\cdot\|$  the associated norm. If  $\alpha$  is an endomorphism of  $\mathcal{P}_n$  with the property that  $\|\alpha(p)\| = \|p\|$  for all  $p \in \mathcal{P}_n$ , then  $\alpha$  is orthogonal in this scalar product.
- The real function  $(p, q) \mapsto (pq)'(0)$ , where  $f'$  is the derivative of  $f$ , defines a scalar product on the subspace  $\{p \in \mathcal{P}_n \mid p(0) = 0\}$ .

## Projections and Reflections

- Let  $v \in \mathbb{R}^n$  be a unit vector and  $Px$  the orthogonal projection of  $x \in \mathbb{R}^n$  in the direction of  $v$ , that is, if  $x = \text{const.} \cdot v$ , then  $Px = x$ , while if  $x \perp v$ , then  $Px = 0$ . Show that  $P = vv^T$  (here  $v^T$  is the transpose of the column vector  $v$ ). In matrix notation,  $(P)_{ij} = v_i v_j$ .
- Continuing, let  $Q$  be the orthogonal projection into the subspace perpendicular to  $v$ . Show that  $Q = I - vv^T$ .

- c) Let  $u$  and  $v$  be orthogonal unit vectors and let  $R$  be the orthogonal projection into the subspace perpendicular to both  $u$  and  $v$ . Show that  $R = I - uu^T - vv^T$ .
115. A linear map  $P: X \rightarrow X$  acting on a vector space  $X$  is called a *projection* if  $P^2 = P$  (this  $P$  is not necessarily an “orthogonal projection”).
- Show that the matrix  $P = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$  is a projection. Draw a sketch of  $\mathbb{R}^2$  showing the vectors  $(1, 2)$ ,  $(-1, 0)$ , and  $(0, 3)$  and their images under the map  $P$ . Also indicate both the image,  $V$ , and nullspace,  $W$ , of  $P$ .
  - Repeat this for  $Q := I - P$ .
  - If the image and nullspace of a projection  $P$  are orthogonal then  $P$  is called an *orthogonal projection*. Let  $M = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$ . For which real value(s) of  $a$  and  $c$  is this a projection? An orthogonal projection?
116. More on general projections, so all one knows is that  $P^2 = P$ .
- Show that in a linear space  $X$ , for a projection  $P$  one can find complementary subspaces  $V$ , and  $W$  so that:  

$$X = V \oplus W, \quad Px = x \text{ for all } x \in V, \quad Px = 0 \text{ for all } x \in W \text{ (so } W \text{ is the nullspace of } P\text{.)}$$
 Thus,  $P$  is the projection onto  $V$ .
  - Show that  $Q := I - P$  is also a projection, but it projects onto  $W$ .
  - If  $P$  is written as a matrix, it is similar to the block matrix  $M = \begin{pmatrix} I_V & 0 \\ 0 & 0_W \end{pmatrix}$ , where  $I_V$  is the identity map on  $V$  and  $0_W$  the zero map on  $W$ .
  - Show that  $\dim \text{image}(P) = \text{trace}(P)$ .
  - If two projections  $P$  and  $Q$  on  $V$  have the same rank, show they are similar.
  - If  $X$  has an inner product, show that the subspaces  $V$  and  $W$  are orthogonal if and only if  $P = P^*$ . Moreover, if  $P = P^*$ , then  $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$ . These are the *orthogonal projections*.
117. ORTHOGONAL PROJECTIONS OF RANK 1 AND  $n - 1$ . Let  $P: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a projection with rank 1.
- Show that  $P$  is an orthogonal projection if and only if there is a unit vector  $v$  such that  $P = vv^T$ , where  $v^T$  is the transpose of  $v$  (see Problem 52).
  - Show that  $Q: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an orthogonal projection with rank  $n - 1$  if and only if there is a unit vector  $v$  such that  $Q = I - vv^T$ , where  $v^T$  is the transpose of  $v$  (so  $v^T$  is a row vector).
  - Let  $Q: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a matrix representing an orthogonal projection. If its diagonal elements are  $5/6$ ,  $2/3$ , and  $1/2$ , find  $Q$  (it is almost uniquely determined).

118. Let  $P$  be a projection, so  $P^2 = P$ . If  $c \neq 1$ , compute  $(I - cP)^{-1}$ .
119. If the matrix  $R$  is a reflection (that is,  $R^2 = I$ ) and  $c \neq \pm 1$  show that  $I - cR$  is invertible by finding an explicit formula for the inverse.
120. A linear map  $R: X \rightarrow X$  acting on a vector space  $X$  is called a *reflection* if  $R^2 = I$ .
- Show that the matrix  $R = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$  is a reflection. Draw a sketch of  $\mathbb{R}^2$  showing the vectors  $(1, 2)$ ,  $(-1, 0)$ , (and  $(0, 3)$ ) and their images under  $R$ . Also indicate both the subspaces  $V$  and  $W$  of vectors that are mapped to themselves:  $Rv = v$ , and those that are mapped to their opposites:  $Rw = -w$ . [From your sketch it is clear that  $V$  and  $W$  are not orthogonal so this  $R$  is not an “orthogonal reflection”.]
  - More generally, show that for any reflection one can write  $X = V \oplus W$  so that  $Rx = x$  for all  $x \in V$  and  $Rx = -x$  for all  $x \in W$ . Thus,  $R$  is the reflection across  $V$ .
  - Show that  $R$  is similar to the block matrix  $M = \begin{pmatrix} I_V & 0 \\ 0 & -I_W \end{pmatrix}$ , where  $I_V$  is the identity map on  $V$ .
  - $X$  has an inner product and the above subspaces  $V$  and  $W$  are orthogonal, then  $R$  is called an *orthogonal reflection*. Let  $S = \begin{pmatrix} -1 & c \\ 0 & 1 \end{pmatrix}$ . For which value(s) of  $c$  is this an orthogonal reflection?
  - Let  $M := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ . For which value(s) of  $a$ ,  $b$ , and  $c$  is  $M$  a reflection? An orthogonal reflection?
  - More generally, show that  $V$  and  $W$  are orthogonal if and only if  $R = R^*$ . This property characterizes an orthogonal reflection.
121. Show that projections  $P$  and reflections  $R$  are related by the formula  $R = 2P - I$ . This makes obvious the relation between the above several problems.
122. Let  $A$  be an  $n \times n$  real matrix with the property that  $(A - \alpha I)(A - \beta I) = 0$ , where  $\alpha \neq \beta$  are real. Show that  $A$  is similar to the block matrix  $\begin{pmatrix} \alpha I_k & 0 \\ 0 & \beta I_{n-k} \end{pmatrix}$ , where  $I_k$  is the  $k \times k$  identity matrix for some appropriate  $k$  (what is it?). [REMARK: This problem generalizes the above Problems 116 and 120 on projections,  $P^2 - P = 0$ , and reflections,  $R^2 - I = 0$ ].
123. Let  $\mathbf{n} := (a, b, c) \in \mathbb{R}^3$  be a *unit* vector and  $\mathcal{S}$  the plane of vectors (through the origin) perpendicular to  $\mathbf{n}$ .

- a) Show that the *orthogonal projection* of  $\mathbf{x}$  in the direction of  $\mathbf{n}$  can be written in the matrix form

$$\langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (\mathbf{n}\mathbf{n}^T)\mathbf{x} = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where  $\langle \mathbf{x}, \mathbf{n} \rangle$  is the usual inner product,  $\mathbf{n}^T$  is the transpose of the column vector  $\mathbf{n}$ , and  $\mathbf{n}\mathbf{n}^T$  is matrix multiplication.

- b) Show that the *orthogonal projection*  $P$  of a vector  $\mathbf{x} \in \mathbb{R}^3$  into  $\mathcal{S}$  is

$$P\mathbf{x} = \mathbf{x} - \langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (I - \mathbf{n}\mathbf{n}^T)\mathbf{x},$$

Apply this to compute the orthogonal projection of the vector  $\mathbf{x} = (1, -2, 3)$  into the plane in  $\mathbb{R}^3$  whose points satisfy  $x - y + 2z = 0$ .

- c) Find a formula similar to the previous part for the orthogonal *reflection*  $R$  of a vector across  $\mathcal{S}$ . Then apply it to compute the orthogonal reflection of the vector  $\mathbf{v} = (1, -2, 3)$  across the plane in  $\mathbb{R}^3$  whose points satisfy  $x - y + 2z = 0$ .
- d) Find a  $3 \times 3$  matrix that projects a vector in  $\mathbb{R}^3$  into the plane  $x - y + 2z = 0$ .
- e) Find a  $3 \times 3$  matrix that reflects a vector in  $\mathbb{R}^3$  across the plane  $x - y + 2z = 0$ .

### Similar Matrices

124. Let  $C$  and  $B$  be square matrices with  $C$  invertible. Show the following.

- a)  $(CBC^{-1})^2 = C(B^2)C^{-1}$
- b) Similarly, show that  $(CBC^{-1})^k = C(B^k)C^{-1}$  for any  $k = 1, 2, \dots$ .
- c) If  $B$  is also invertible, is it true that  $(CBC^{-1})^{-2} = C(B^{-2})C^{-1}$ ? Why?

125. Let  $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$ .

- a) Find an invertible matrix  $C$  such that  $D := C^{-1}AC$  is a diagonal matrix. Thus,  $A = CDC^{-1}$ .
- b) Compute  $A^{50}$ .

126. Let  $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , and  $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,

- a) Are  $A$  and  $B$  similar? Why?
- b) Show that  $B$  is not similar to any diagonal matrix.

127. a) Show that the matrices  $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  are similar.
- b) Let  $A(s) = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$  and let  $M = A(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . If  $s \neq 0$ , show that  $A(s)$  is similar to  $M$ .

REMARK: This is a simple and fundamental counterexample to the assertion: "If  $A(s)$  depends smoothly on the parameter  $s$  and is similar to  $M$  for all  $s \neq 0$ , then  $A(0)$  is also similar to  $M$ ."

128. Say a matrix  $A$  is *similar* to the matrix  $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ . Give a proof or counterexample for each of the following assertions.

- |                             |   |
|-----------------------------|---|
| a). $A^2 = A$               | d). $\lambda = 0$ is an eigenvalue of $A$ . |
| b). $\det A = 0$ .          | e). $\lambda = 1$ is an eigenvalue of $A$ . |
| c). $\text{trace } A = 1$ . | f). $v = (1, 0)$ is an eigenvector of $A$ . |

129. Let  $A$  be a square real matrix. For each of the following assertions, either give a proof or find a counterexample.

- If  $A$  is similar to the identity matrix, then  $A = I$ .
- If  $A$  is similar to the zero matrix, then  $A = 0$ .
- If  $A$  is similar to  $2A$ , then  $A = 0$ .
- If all the eigenvalues of  $A$  are zero, then  $A = 0$ .
- If  $A$  is similar to a matrix  $B$  with the property  $B^2 = 0$ , then  $A^2 = 0$ .
- If  $A$  is similar to a matrix  $B$  one of whose eigenvalues is 7, then one eigenvalue of  $A$  is 7.
- If  $A$  is similar to a matrix  $B$  that can be diagonalized, then  $A$  can be diagonalized.
- If  $A$  can be diagonalized and  $A^2 = 0$ , then  $A = 0$ .
- If  $A$  is similar to a projection  $P$  (so  $P^2 = P$ ), then  $A$  is a projection.
- If  $A$  is similar to a real orthogonal matrix, then  $A$  is an orthogonal matrix.
- If  $A$  is similar to a symmetric matrix, then  $A$  is a symmetric matrix.

130. A square matrix  $M$  is *diagonalized* by an invertible matrix  $S$  if  $SMS^{-1}$  is a diagonal matrix. Of the following three matrices, one can be diagonalized by an orthogonal matrix, one can be diagonalized but not by any orthogonal matrix, and one cannot be diagonalized. State which is which — and why.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix}.$$

131. Repeat the previous problem for the matrices

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

132. Let  $A$  be the matrix

$$A = \begin{pmatrix} 1 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 1 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 1 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \lambda \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Show that there exists a matrix  $B$  with  $BAB^{-1} = A^T$  (here  $A^T$  is the transpose of  $A$ ).

133. Let  $A$  be an  $n \times n$  matrix with coefficients in a field  $\mathbb{F}$  and let  $S$  be an invertible matrix.

- If  $SAS^{-1} = \lambda A$  for some  $\lambda \in \mathbb{F}$ , show that either  $\lambda^n = 1$  or  $A$  is nilpotent.
- If  $n$  is odd and  $SAS^{-1} = -A$ , show that  $0$  is an eigenvalue of  $A$ .
- If  $n$  is odd and  $SAS^{-1} = A^{-1}$ , show that  $1$  is an eigenvalue of  $A$ .

134. Let  $A(t) = \begin{pmatrix} 1+t & 1 \\ -t^2 & 1-t \end{pmatrix}$ .

- Show that  $A(t)$  is similar to  $A(0)$  for all  $t$ .
- Show that  $B(t) := A(0) + A'(0)t$  is similar to  $A(0)$  only for  $t = 0$ .

135. Let  $f$  be any function defined on  $n \times n$  matrices with the property that  $f(AB) = f(BA)$  [Example:  $f(A) = \text{trace}(A)$ ]. If  $A$  and  $C$  are similar, show that  $f(A) = f(C)$ .

136. Let  $h(A)$  be a scalar-valued function defined on all square matrices  $A$  having the property that if  $A$  and  $B$  are similar, then  $h(A) = h(B)$ . If  $h$  is also linear, show that  $h(A) = c \text{trace}(A)$  where  $c$  is a constant.

137. Let  $\{A, B, C, \dots\}$  be linear maps over a finite dimensional vector space  $V$ . Assume these matrices all commute pairwise, so  $AB = BA$ ,  $AC = CA$ ,  $BC = CB$ , etc.

- Show that there is some basis for  $V$  in which all of these are represented simultaneously by upper triangular matrices.

- b) If each of these matrices can be diagonalized, show that there is some basis for  $V$  in which all of these are represented simultaneously by diagonal matrices.

### Symmetric and Self-adjoint Maps

138. Proof or Counterexample. Here  $A$  is a real symmetric matrix.

- Then  $A$  is invertible.
- $A$  is invertible if and only if  $\lambda = 0$  is *not* an eigenvalue of  $A$ .
- The eigenvalues of  $A$  are all real.
- If  $A$  has eigenvectors  $v, w$  corresponding to eigenvalues  $\lambda, \mu$  with  $\lambda \neq \mu$ , then  $\langle v, w \rangle = 0$ .
- If  $A$  has linearly independent eigenvectors  $v$  and  $w$  then  $\langle v, w \rangle = 0$ .
- If  $B$  is any square real matrix, then  $A := B^*B$  is positive semi-definite.
- If  $B$  is any square matrix, then  $A := B^*B$  is positive definite if and only if  $B$  is invertible.
- If  $C$  is a real anti-symmetric matrix (so  $C^* = -C$ ), then  $\langle v, Cv \rangle = 0$  for all real vectors  $v$ .

139. Let  $A$  and  $B$  be symmetric matrices with  $A$  positive definite.

- Show there is a change of variables  $y = Sx$  (so  $S$  is an invertible matrix) so that  $\langle x, Ax \rangle = \|y\|^2$  (equivalently,  $S^TAS = I$ ). One often rephrases this by saying that a positive definite matrix is *congruent* to the identity matrix.
- Show there is a linear change of variables  $y = Px$  so that *both*  $\langle x, Ax \rangle = \|y\|^2$  and  $\langle x, Bx \rangle = \langle y, Dy \rangle$ , where  $D$  is a diagonal matrix.
- If  $A$  is a positive definite matrix and  $B$  is positive semi-definite, show that

$$\text{trace}(AB) \geq 0$$

with equality if and only if  $B = 0$ .

140. [Congruence of Matrices] Two symmetric matrices  $A, B$  in  $M(n, \mathbb{F})$  are called *congruent* if there is an invertible matrix  $T \in M(n, \mathbb{F})$  with  $A = T^*BT$  (here  $T^*$  is the hermitian adjoint of  $T$ ); equivalently, if

$$\langle Tx, ATy \rangle = \langle X, BY \rangle \quad \text{for all vectors } x, y,$$

so  $T$  is just a change of coordinates.

True or False?

- a) Over  $\mathbb{R}$  the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  is congruent to  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .
- b) If  $A$  and  $B$  are congruent over  $\mathbb{C}$ , then  $A$  and  $B$  are similar over  $\mathbb{C}$ .
- c) If  $A$  is real and all of its eigenvalues are positive, then over  $\mathbb{R}$   $A$  is congruent to the identity matrix.
- d) Over  $\mathbb{R}$  if  $A$  is congruent to the identity matrix, then all of its eigenvalues are positive.

141. Let  $A$  be an  $n \times n$  real symmetric matrix with eigenvalues  $\lambda_1 \leq \dots \leq \lambda_n$  and corresponding orthonormal eigenvectors  $v_1, \dots, v_n$ .

a) Show that

$$\lambda_1 = \min_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2} \quad \text{and} \quad \lambda_n = \max_{x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2}.$$

b) Show that

$$\lambda_2 = \min_{x \perp v_1, x \neq 0} \frac{\langle x, Ax \rangle}{\|x\|^2}.$$

142. Let  $A = (a_{ij})$  be an  $n \times n$  real symmetric matrix with eigenvalues  $\lambda_1 \leq \dots \leq \lambda_n$  and let  $C = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$  be the upper-left  $2 \times 2$  block of  $A$  with eigenvalues  $\mu_1 \leq \mu_2$ .

- a) Show that  $\lambda_1 \leq \mu_1$  and  $\lambda_n \geq \mu_2$ .
- b) Generalize.

143. Let  $M = (m_{ij})$  be a real symmetric  $n \times n$  matrix and let  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Further, let  $Q(x)$  be the quadratic polynomial

$$Q(x) = \sum_{i,j} m_{ij} x_i x_j.$$

In terms of the rank and signature of  $M$ , give a necessary and sufficient condition that the set  $\{x \in \mathbb{R}^n \mid Q(x) = 1\}$  is bounded and non-empty.

144. Suppose that  $A$  is a real  $n \times n$  symmetric matrix with two equal eigenvalues. If  $v$  is any vector, show that the vectors  $v, Av, \dots, A^{n-1}v$  are linearly dependent.

145. Let  $A$  be a positive definite  $n \times n$  matrix with diagonal elements  $a_{11}, a_{22}, \dots, a_{nn}$ . Show that

$$\det A \leq \prod a_{ii}.$$

146. Let  $A$  be a positive definite  $n \times n$  matrix. Show that  $\det A \leq \left(\frac{\text{trace } A}{n}\right)^n$ . When can equality occur?
147. Let  $Q$  and  $M$  be symmetric matrices with  $Q$  invertible. Show there is a matrix  $A$  such that  $AQ + QA^* = M$ .
148. Let the real matrix  $A$  be *anti-symmetric* (or *skew-symmetric*), that is,  $A^* = -A$ .
- Give an example of a  $2 \times 2$  anti-symmetric matrix.
  - Show that the diagonal elements of any  $n \times n$  anti-symmetric matrix must all be zero.
  - Show that every square matrix can (uniquely?) be written as the sum of a symmetric and an anti-symmetric matrix.
  - Show that the eigenvalues of a real anti-symmetric matrix are purely imaginary.
  - Show that  $\langle \mathbf{V}, A\mathbf{V} \rangle = 0$  for every vector  $\mathbf{V}$ .
  - If  $A$  is an  $n \times n$  anti-symmetric matrix and  $n$  is odd, show that  $\det A = 0$  — and hence deduce that  $A$  cannot be invertible.
  - If  $n$  is even, show that  $\det A \geq 0$ . Show by an example that  $A$  may be invertible.
  - If  $A$  is a real invertible  $2n \times 2n$  anti-symmetric matrix, show there is a real invertible matrix  $S$  so that

$$A = SJS^*,$$

where  $J := \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$ ; here  $I_k$  is the  $k \times k$  identity matrix. [Note that  $J^2 = -I$  so the matrix  $J$  is like the complex number  $i = \sqrt{-1}$ .

## Orthogonal and Unitary Maps

149. Let the real  $n \times n$  matrix  $A$  be an *isometry*, that is, it preserves length:

$$\|Ax\| = \|x\| \quad \text{for all vectors } x \in \mathbb{R}^n. \quad (1)$$

These are the *orthogonal transformations*.

- a) Show that (1) is equivalent to  $\langle Ax, Ay \rangle = \langle x, y \rangle$  for all vectors  $x, y$ , so  $A$  preserves inner products. HINT: use the *polarization* identity:

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2). \quad (2)$$

This shows how, in a real vector space, to recover a the inner product if you only know how to compute the (euclidean) length.

- b) Show that (1) is equivalent to  $A^{-1} = A^*$ .
- c) Show that (1) is equivalent to the columns of  $A$  being unit vectors that are mutually orthogonal.
- d) Show that (1) implies  $\det A = \pm 1$  and that all eigenvalues satisfy  $|\lambda| = 1$ .
- e) If  $n = 3$  and  $\det A = +1$ , show that  $\lambda = 1$  is an eigenvalue.
- f) Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  have the property (1), namely  $\|F(x)\| = \|x\|$  for all vectors  $x \in \mathbb{R}^n$ . Then  $F$  is an orthogonal transformation. Proof or counterexample.
- g) Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a *rigid motion*, that is, it preserves the distance between any two points:  $\|F(x) - F(y)\| = \|x - y\|$  for all vectors  $x, y \in \mathbb{R}^n$ . Show that  $F(x) = F(0) + Ax$  for some orthogonal transformation  $A$ .

150. Let  $\mathbf{n} \in \mathbb{R}^3$  be a unit vector. Find a formula for the  $3 \times 3$  matrix that determines a rotation of  $\mathbb{R}^3$  through an angle  $\theta$  with  $\mathbf{n}$  as axis of rotation (assuming the axis passes through the origin). Here we outline one approach to find this formula — but before reading further, try finding it on your own.

- a) (Example) Find a matrix that rotates  $\mathbb{R}^3$  through the angle  $\theta$  using the vector  $(1, 0, 0)$  as the axis of rotation.
- b) More generally, let  $\mathbf{u}$  and  $\mathbf{w}$  be orthonormal vectors in the plane perpendicular to  $\mathbf{n}$ . Show that the map

$$R_{\mathbf{n}} : \mathbf{x} \mapsto (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos \theta \mathbf{u} + \sin \theta \mathbf{w}$$

rotates  $\mathbf{x}$  through an angle  $\theta$  with  $\mathbf{n}$  as axis of rotation. [Note: one needs more information to be able to distinguish between  $\theta$  and  $-\theta$ ].

- c) Using Problem 123 to write  $\mathbf{u}$  and  $\mathbf{w}$ , in terms of  $\mathbf{n}$  and  $\mathbf{x}$ , show that one can rewrite the above formula as

$$\begin{aligned} R_{\mathbf{n}}\mathbf{x} &= (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos \theta [\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}] + \sin \theta (\mathbf{n} \times \mathbf{x}) \\ &= \mathbf{x} + \sin \theta (\mathbf{n} \times \mathbf{x}) + (1 - \cos \theta)[(\mathbf{x} \cdot \mathbf{n})\mathbf{n} - \mathbf{x}]. \end{aligned}$$

Thus, using Problem 49, if  $\mathbf{n} = (a, b, c) \in \mathbb{R}^3$  deduce that:

$$R_{\mathbf{n}} = I + \sin \theta \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} + (1 - \cos \theta) \begin{pmatrix} -b^2 - c^2 & ab & ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{pmatrix}.$$

- d) Let  $A_{\mathbf{n}}$  be as in Problem 49 (but using  $\mathbf{n}$  rather than  $\mathbf{v}$ ). Show that

$$R_{\mathbf{n}} = I + \sin \theta A_{\mathbf{n}} + (1 - \cos \theta) A_{\mathbf{n}}^2$$

(see more on this in Problem 167).

- e) Use this formula to find the matrix that rotates  $\mathbb{R}^3$  through an angle of  $\theta$  using as axis the line through the origin and the point  $(1, 1, 1)$ .

151. Recall (see Problem 123) that  $\mathbf{u} := \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}$  is the projection of  $\mathbf{x}$  into the plane perpendicular to the unit vector  $\mathbf{n}$ . Show that in  $\mathbb{R}^3$  the vector

$$\mathbf{w} := \mathbf{n} \times \mathbf{u} = \mathbf{n} \times [\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}] = \mathbf{n} \times \mathbf{x}$$

is perpendicular to both  $\mathbf{n}$  and  $\mathbf{u}$ , and that  $\mathbf{w}$  has the same length as  $\mathbf{u}$ . Thus  $\mathbf{n}$ ,  $\mathbf{u}$ , and  $\mathbf{w}$  are orthogonal with  $\mathbf{u}$ , and  $\mathbf{w}$  in the plane perpendicular to the axis of rotation  $\mathbf{n}$ .

152. a) Let  $V$  be a complex vector space and  $A : V \rightarrow V$  a unitary operator. Show that  $A$  is diagonalizable.  
 b) Does the same remain true if  $V$  is a real vector space, and  $A$  is orthogonal?

153. For a complex vector space with a hermitian inner product one can define a unitary matrix  $U$  just as in Problem 149 as one that preserves the length:

$$\|Uv\| = \|v\|$$

for all complex vectors  $v$ .

- a) In this situation, for any complex vectors  $u, v$  prove the *polarization identity*

$$\langle u, v \rangle = \frac{1}{4} [(\|u + v\|^2 - \|u - v\|^2) + i(\|u + iv\|^2 - \|u - iv\|^2)].$$

- b) Extend Problem 149 to unitary matrices.

154. Show that the only real matrix that is orthogonal, symmetric and positive definite is the identity.

155. Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$  and  $W$  a finite dimensional vector space over  $\mathbb{C}$ .

True or False

- a) Let  $\alpha$  be an endomorphism of  $W$ . In a unitary basis for  $W$  say  $M$  is a diagonal matrix all of whose eigenvalues satisfy  $|\lambda| = 1$ . Then  $\alpha$  is a unitary matrix.  
 b) The set of orthogonal endomorphisms of  $V$  forms a ring under the usual addition and multiplication.  
 c) Let  $\alpha \neq I$  be an orthogonal endomorphism of  $V$  with determinant 1. Then there is no  $v \in V$  (except  $v = 0$ ) satisfying  $\alpha(v) = v$ .

- d) Let  $\alpha$  be an orthogonal endomorphism of  $V$  and  $\{v_1, \dots, v_k\}$  a linearly independent set of vectors in  $V$ . Then the vectors  $\{\alpha(v_1), \dots, \alpha(v_k)\}$  are linearly independent.
- e) Using the standard scalar product for  $\mathbb{R}^3$ , let  $v \in \mathbb{R}^3$  be a unit vector,  $\|v\| = 1$ , and define the endomorphism  $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  using the cross product:  $\alpha(x) := v \times x$ . Then the subspace  $v^\perp$  is an invariant subspace of  $\alpha$  and  $\alpha$  is an orthogonal map on this subspace.

156. Let  $R(t)$  be a family of real orthogonal matrices that depend smoothly on the real parameter  $t$ .

- a) If  $R(0) = I$ , show that the derivative,  $A := R'(0)$  is anti-symmetric, that is,  $A^* = -A$ .
- b) Let the vector  $x(t)$  be a solution of the differential equation  $x' = A(t)x$ , where the matrix  $A(t)$  is anti-symmetric. Show that its (Euclidean) length is constant,  $\|x(t)\| = \text{const}$ . In other words, using this  $x(t)$  if we define the map  $R(t)$  by  $R(t)x(0) := x(t)$ , then  $R(t)$  is an orthogonal transformation.
- c) Let  $A(t)$  be an anti-symmetric matrix and let the square matrix  $R(t)$  satisfy the differential equation  $R' = AR$  with  $R(0)$  an orthogonal matrix. Show that  $R(t)$  is an orthogonal matrix.

### Normal Matrices

157. A square matrix  $M$  is called *normal* if it commutes with its adjoint:  $AA^* = A^*A$ . For instance all self-adjoint and all orthogonal matrices are normal.

- a) Give an example of a normal matrix that is neither self-adjoint nor orthogonal.
- b) Show that  $M$  is normal if and only if  $\|MX\| = \|M^*X\|$  for all vectors  $X$ .
- c) Let  $M$  be normal and  $V$  an eigenvector with eigenvalue  $\lambda$ . Show that  $V$  is also an eigenvector of  $M^*$ , but with eigenvalue  $\bar{\lambda}$ . [SUGGESTION: Notice that  $L := M - \lambda I$  is also normal.]
- d) If  $M$  is normal, show that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

158. Here  $A$  and  $B$  are  $n \times n$  complex matrices.

True or False

- a) If  $A$  is normal and  $\det(A) = 1$ , then  $A$  is unitary.
- b) If  $A$  is unitary, then  $A$  is normal and  $\det(A) = 1$ .
- c) If  $A$  is normal and has real eigenvalues, then  $A$  is hermitian (that is, self-adjoint).
- d) If  $A$  and  $B$  are hermitian, then  $AB$  is normal.

- e) If  $A$  is normal and  $B$  is unitary, then  $\bar{B}^T AB$  is normal.

## Symplectic Maps

159. Let  $B$  be a real  $n \times n$  matrix with the property that  $B^2 = -I$ .

- Show that  $n$  must be even,  $n = 2k$ .
- Show that  $B$  is similar to the block matrix  $J := \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$ , where here  $I_k$  is the  $k \times k$  identity matrix. [HINT: Write  $x_1 := \frac{1}{2}(I - B)x$  and  $x_2 := \frac{1}{2}(I + B)x$ . Note that  $x_1 + x_2 = x$ . Compute  $Bx = ?$ ].
- Let  $C$  be a real  $n \times n$  matrix with the property that  $(C - \lambda I)(C - \bar{\lambda}I) = 0$ , where  $\lambda = \alpha + i\beta$  with  $\alpha$  and  $\beta$  real and  $\beta \neq 0$ . Show that  $C$  is similar to the matrix  $K := \alpha I + \beta J$  with  $J$  as above.

160. Let  $J := \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$ , where  $I_k$  is the  $k \times k$  identity matrix. Note that  $J^2 = -I$ . A real  $2k \times 2k$  matrix  $S$  is *symplectic* if it preserves the bilinear form  $B[x, y] := \langle x, Jy \rangle$ ; thus  $B[Sx, Sy] = B[x, y]$  for all vectors  $x, y$  in  $\mathbb{R}^{2k}$ .

- Is  $J$  itself symplectic?
- Show that a symplectic matrix is invertible and that the inverse is also symplectic.
- Show that the set  $Sp(2k)$  of  $2k \times 2k$  symplectic matrices forms a group. [In many ways this is analogous to the orthogonal group].
- Show that a matrix  $S$  is symplectic if and only if  $S^*JS = J$ . Then deduce that  $S^{-1} = -JS^*J$  and that  $S^*$  is also symplectic.
- Show that if  $S$  is symplectic, then  $S^*$  is similar to  $S^{-1}$ . Thus, if  $\lambda$  is an eigenvalue of  $S$ , then so are  $\bar{\lambda}$ ,  $1/\lambda$ , and  $1/\bar{\lambda}$ .
- Write a symplectic matrix  $S$  have the block form  $S := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , where  $A, B, C$ , and  $D$  are  $k \times k$  real matrices. Show that  $S$  is symplectic if and only if

$$A^*C = C^*A, \quad B^*D = D^*B, \quad \text{and} \quad A^*D - C^*B = I.$$

Show that

$$S^{-1} = \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix}.$$

- If  $S$  is symplectic, show that  $\det S = +1$ . One approach is to use the previous part, picking the block matrices  $X$  and  $Y$  so that

$$\begin{pmatrix} I & 0 \\ X & Y \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}.$$

- h) Let  $S(t)$  be a family of real symplectic matrices that depend smoothly on the real parameter  $t$  with  $S(0) = I$ . Show that the derivative  $T := S'(0)$  has the property that  $JT$  is self-adjoint.
- i) Let the matrix  $S(t)$  be a solution of the differential equation  $S'(t) = TS$  with  $S(0)$  a symplectic matrix, where  $T$  is a real square matrix with the property that  $JT$  is self-adjoint. Show that  $S(t)$  is a symplectic matrix.

## Least Squares

161. Find the straight line  $y = a + mx$  that is the best least squares fit to the points  $(0, 0)$ ,  $(1, 3)$ , and  $(2, 7)$ .
162. Let  $L : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be a linear map. If the equation  $Lx = b$  has no solution, instead frequently one wants to pick  $x$  to minimize the error:  $\|Lx - b\|$  (here we use the Euclidean distance). Assume that the nullspace of  $L$  is zero.
- a) Show that the desired  $x$  is a solution of the *normal equations*  $L^*Lx = L^*b$  (here  $L^*$  is the adjoint of  $L$ ). Note that since the nullspace of  $L$  is zero,  $L^*L : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is invertible (why?).
- b) Apply this to find the optimal horizontal line that fits the three data points  $(0, 1)$ ,  $(1, 2)$ ,  $(4, 3)$ .
- c) Similarly, find the optimal straight line (not necessarily horizontal) that fits the same data points.
163. Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be a linear map. If  $A$  is not one-to-one, but the equation  $Ax = y$  has some solution, then it has many. Is there a “best” possible answer? What can one say? Think about this before reading the next paragraph.

If there is some solution of  $Ax = y$ , show there is exactly one solution  $x_1$  of the form  $x_1 = A^*w$  for some  $w$ , so  $AA^*w = y$ . Moreover of all the solutions  $x$  of  $Ax = y$ , show that  $x_1$  is closest to the origin (in the Euclidean distance). [REMARK: This situation is related to the case where where  $A$  is not onto, so there may not be a solution — but the method of least squares gives an “best” approximation to a solution.]

164. Let  $P_1, P_2, \dots, P_k$  be  $k$  points (think of them as *data*) in  $\mathbb{R}^3$  and let  $\mathcal{S}$  be the plane

$$\mathcal{S} := \{X \in \mathbb{R}^3 : \langle X, N \rangle = c\},$$

where  $N \neq 0$  is a unit vector normal to the plane and  $c$  is a real constant. This problem outlines how to find the plane that *best approximates the data points* in the sense that

it minimizes the function

$$Q(N, c) := \sum_{j=1}^k \text{distance}(P_j, \mathcal{S})^2.$$

a) Show that for a given point  $P$

$$\text{distance}(P, \mathcal{S}) = |\langle X, N \rangle - c| = |\langle X, N \rangle - c|.$$

b) First do the special case where the center of mass  $\bar{P} := \frac{1}{k} \sum_{j=1}^k P_j$  is at the origin,  $\bar{P} = 0$ , showing the plane is determined by letting  $N$  be an eigenvector of the matrix

$$A := \sum_{j=1}^k P_j P_j^T$$

corresponding to it's lowest eigenvalue (here we view the  $P_j$  as column vectors,  $P_j^T$  is the transpose so  $P_j P_j^T$  is a square matrix). What is  $c$ ?

- c) Reduce the general case to the previous case by letting  $V_j = P_j - \bar{P}$ .
- d) Find the equation of the line  $ax + by = c$  that, in the above sense, best fits the data points  $(-1, 3)$ ,  $(0, 1)$ ,  $(1, -1)$ ,  $(2, -3)$ .
- e) Let  $P_j := (p_{j1}, \dots, p_{j3})$ ,  $j = 1, \dots, k$  be the coordinates of the  $j^{\text{th}}$  data point and  $Z_\ell := (p_{1\ell}, \dots, p_{k\ell})$ ,  $\ell = 1, \dots, 3$  be the vector of  $\ell^{\text{th}}$  coordinates. If  $a_{ij}$  is the  $ij$  element of  $A$ , show that  $a_{ij} = \langle Z_i, Z_j \rangle$ . Note that this exhibits  $A$  as a *Gram matrix* (see Problem 107).
- f) Generalize to where  $P_1, P_2, \dots, P_k$  are  $k$  points in  $\mathbb{R}^n$ .

## The Exponential Map

165. For any square matrix  $A$ , define the *exponential*,  $e^A$ , by the usual power series

$$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

- a) Show that the series always converges.
- b) Show that  $e^{(s+t)A} = e^{sA} e^{tA}$  for all real or complex  $s, t$ .
- c) If  $AB = BA$ , show that  $e^{A+B} = e^A e^B$ . Give an example showing this is false if  $A$  and  $B$  don't commute.
- d) If  $A$  is a  $3 \times 3$  diagonal matrix, compute  $e^A$ .

e) If  $A^2 = 0$ , compute  $e^A$ .

f) Compute  $e^A$  for the matrix  $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ .

g) If  $P$  is a projection (so  $P^2 = P$ ) and  $t \in \mathbb{R}$ , compute  $e^{tP}$ .

h) If  $R$  is a reflection (so  $R^2 = I$ ) and  $t \in \mathbb{R}$ , compute  $e^{tR}$ .

i) For real  $t$  show that

$$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

(The matrix on the right is a rotation of  $\mathbb{R}^2$  through the angle  $t$ ).

j) If a (square) matrix  $A$  satisfies  $A^2 = \alpha^2 I$ , show that

$$e^A = \cosh \alpha I + \frac{\sinh \alpha}{\alpha} A.$$

k) If a square matrix  $A$  satisfies  $A^3 = \alpha^2 A$  for some real or complex  $\alpha$ , show that

$$e^A = I + \frac{\sinh \alpha}{\alpha} A + \frac{\cosh \alpha - 1}{\alpha^2} A^2.$$

This formula generalizes the previous part. What is the formula if  $A^3 = -\alpha^2 A$ ?

166. If  $A$  is a diagonal matrix, show that

$$\det(e^A) = e^{\text{trace}(A)}.$$

Is this formula valid for any matrix, not just a diagonal matrix?

167. a) Let  $\mathbf{v} = (\alpha, \beta, \gamma)$  be any non-zero vector. Using the matrix notation from Problem 49, show that

$$A_{\mathbf{v}}^3 = -|\mathbf{v}|^2 A_{\mathbf{v}}.$$

b) Use this (and the definition of  $e^A$  from Problem 165) to verify that

$$e^{A_{\mathbf{v}}} = I + \frac{\sin |\mathbf{v}|}{|\mathbf{v}|} A_{\mathbf{v}} + \frac{1 - \cos |\mathbf{v}|}{|\mathbf{v}|^2} A_{\mathbf{v}}^2.$$

Thus  $e^{A_{\mathbf{v}}}$  is the same as  $R_{\mathbf{n}}$  of Problem 150, where  $\mathbf{n} = \mathbf{v}/|\mathbf{v}|$  and you interpret  $|\mathbf{v}|$  as the angle of rotation  $\theta$ . [See Duistermaat and Kolk, *Lie Groups*, Section 1.4 for an explanation. There the anti-symmetric matrix  $A_{\mathbf{v}}$  is viewed as an element of the Lie algebra associated with the Lie group of  $3 \times 3$  orthogonal matrices.]

168. a) If  $A$  is a constant matrix (so it does not depend on  $t$ ), compute the derivative of  $e^{tA}$  with respect to the real parameter  $t$ .
- b) If  $M := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ , find a constant matrix  $A$  so that  $M = e^{tA}$  for all real  $t$ .
- c) If  $N := \begin{pmatrix} \cosh t & -\sinh t \\ \sinh t & \cosh t \end{pmatrix}$ , show there is *no* constant matrix  $A$  so that  $N = e^{tA}$ .

169. Let  $A$  be a square constant matrix. Show that the (unique) solution  $X(t)$  of the matrix differential equation

$$\frac{dX(t)}{dt} = AX(t), \quad \text{with } X(0) = I$$

is  $X(t) = e^{tA}$ .

### Jordan Form

170. [Jordan Normal Form] Let

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 \end{pmatrix},$$

- a) In the Jordan normal form for  $A$ , how often does the largest eigenvalue of  $A$  occur on the diagonal?
- b) For  $A$ , find the dimension of the eigenspace corresponding to the eigenvalue 4.

171. Determine the Jordan normal form of

$$B := \begin{pmatrix} -3 & -2 & 0 \\ 4 & 3 & 0 \\ 2 & 1 & -1 \end{pmatrix}.$$

### Derivatives

172. Let  $A(t) = (a_{ij}(t))$  be a square real matrix whose elements are smooth functions of the real variable  $t$  and write  $A'(t) = (a'_{ij}(t))$  for the matrix of derivatives. [There is an obvious equivalent coordinate-free definition of the derivative of a matrix using  $\lim_{h \rightarrow 0} [A(t+h) - A(t)]/h$ .]

- Compute the derivative:  $dA^3(t)/dt$ .
- If  $A(t)$  is invertible, find the formula for the derivative of  $A^{-1}(t)$ . Of course it will resemble the  $1 \times 1$  case  $-A'(t)/A^2(t)$ .

173. Let  $A(t)$  be a square real matrix whose elements are smooth functions of the real variable  $t$ . Assume  $\det A(t) > 0$ .

- Show that  $\frac{d}{dt} \log \det A = \text{trace}(A^{-1}A')$ .
- Conclude that for any invertible matrix  $A(t)$

$$\frac{d \det A(t)}{dt} = \det A(t) \text{trace}[A^{-1}(t)A'(t)].$$

- If  $\det A(t) = 1$  for all  $t$  and  $A(0) = I$ , show that the matrix  $A'(0)$  has trace zero.
- Compute:  $\frac{d^2}{dt^2} \log \det A(t)$ .

### Block Matrices

The next few problems illustrate the use of block matrices. (See also Problems 68, 116, 120, 159, and 160.)

174. Let  $M = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$  be an  $(n+k) \times (n+k)$  block matrix partitioned into the

$n \times n$  matrix  $A$ , the  $n \times k$  matrix  $B$ , the  $k \times n$  matrix  $C$  and the  $k \times k$  matrix  $D$ .

- Show that matrices of the above form but with  $C = 0$  are a ring.
- If  $C = 0$ , find a formula for  $M^{-1}$  involving  $A^{-1}$ , etc.
- If  $B = 0$  and  $C = 0$ , show that  $\det M = (\det A)(\det D)$ . [SUGGESTION: One approach begins with  $M = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}$  for some appropriate matrix  $X$ .]
- If  $B = 0$  or  $C = 0$ , show that  $\det M = \det A \det D$ . [SUGGESTION: If  $C = 0$ , observe that  $\det M = \det \left[ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \right]$  and pick the matrix  $X$  cleverly.]
- If  $A$  is invertible, show that  $\det M = \det A \det(D - CA^{-1}B)$ . If  $M$  is  $2 \times 2$ , this of course reduces to  $ad - bc$ . There is of course a similar formula only assuming  $D$  is invertible.

175. Let  $M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$  be a square block matrix, where  $A$  is also a square matrix.
- Find the relation between the non-zero eigenvalues of  $M$  and those of  $A$ . What about the corresponding eigenvectors?
  - Proof or Counterexample:  $M$  is diagonalizable if and only if  $A$  is diagonalizable.

176. If a unitary matrix  $M$  has the block form  $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ , show that  $B = 0$  while both  $A$  and  $D$  must themselves be unitary.

177. Let the square matrix  $M$  have the block form  $M := \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ .

- If  $B$  and  $C$  are square, show that  $M$  is invertible if and only if both  $B$  and  $C$  are, and find an explicit formula for  $M^{-1}$ . [ANSWER:  $M^{-1} := \begin{pmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{pmatrix}$ ].
- Even if  $B$  and  $C$  are *not* square but both  $A$  and  $T := CA^{-1}B$  are invertible, find a formula for  $M^{-1}$  of the form  $M^{-1} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ . [ANSWER:

$$M^{-1} = \begin{pmatrix} A^{-1} - A^{-1}BT^{-1}CA^{-1} & A^{-1}BT^{-1} \\ T^{-1}CA^{-1} & -T^{-1} \end{pmatrix} ] .$$

178. Let  $L : V \rightarrow V$  be a linear map acting on the finite dimensional linear vector space mapping  $V$  and say for some subspace  $U \in V$  we have  $L : U \rightarrow U$ , so  $U$  is an  $L$ -invariant subspace. Pick a basis for  $U$  and extend it to a basis for  $V$ . If in this basis  $A : U \rightarrow U$  is the square matrix representing the action of  $L$  on  $U$ , show that in this basis the matrix representing  $L$  on  $V$  has the block matrix form

$$\begin{pmatrix} A & * \\ 0 & * \end{pmatrix},$$

where  $0$  is a matrix of zeroes having the same number of columns as the dimension of  $U$  and  $*$  represent other matrices.

## Interpolation

179. a) Find a cubic polynomial  $p(x)$  with the properties that  $p(0) = 1$ ,  $p(1) = 0$ ,  $p(3) = 2$ , and  $p(4) = 5$ . Is there more than one such polynomial?
- b) Given any points  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$ ,  $(x_4, y_4)$  with the  $x_i$ 's distinct, show there is a unique cubic polynomial  $p(x)$  with the properties that  $p(x_i) = y_i$ .

180. Let  $a_0, a_1, \dots, a_n$  be  $n + 1$  distinct real numbers and  $b_0, b_1, \dots, b_n$  be given real numbers. One seeks an *interpolating polynomial*  $p(x) = c_k x^k + \dots + c_1 x + c_0$  that passes through these points  $(a_j, b_j)$ ; thus  $p(a_j) = b_j, j = 0, \dots, n$ .
- If  $k = n$  show there exists such a polynomial and that it is unique.
  - If  $p$  has the special form  $p(x) = c_{n+1} x^{n+1} + \dots + c_1 x$  (so  $k = n + 1$  and  $c_0 = 0$ ), discuss both the existence and uniqueness of such a polynomial.
  - If  $p$  has the special form  $p(x) = x^{n+1} + c_n x^n + \dots + c_1 x + c_0$ , discuss both the existence and uniqueness of such a polynomial.

### Miscellaneous Problems

181. A *tridiagonal matrix* is a square matrix with zeroes everywhere except on the main diagonal and the diagonals just above and below the main diagonal.

Let  $T$  be a real anti-symmetric tridiagonal matrix with elements  $t_{12} = c_1, t_{23} = c_2, \dots, t_{n-1n} = c_{n-1}$ . If  $n$  is even, compute  $\det T$ .

182. [DIFFERENCE EQUATIONS] One way to solve a second order linear difference equation of the form  $x_{n+2} = ax_n + bx_{n+1}$  where  $a$  and  $b$  are constants is as follows. Let  $u_n := x_n$  and  $v_n := x_{n+1}$ . Then  $u_{n+1} = v_n$  and  $v_{n+1} = au_n + bv_n$ , that is,

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

which, in obvious matrix notation, can be written as  $U_{n+1} = AU_n$ . Consequently,  $U_n = A^n U_0$ . If one can diagonalize  $A$ , the problem is then straightforward. Use this approach to find a formula for the Fibonacci numbers  $x_{n+2} = x_n + x_{n+1}$  with initial conditions  $x_0 = 0$  and  $x_1 = 1$ .

183. Let  $\mathcal{P}$  be the vector space of all polynomials with real coefficients. For any fixed real number  $t$  we may define a linear functional  $L$  on  $\mathcal{P}$  by  $L(f) = f(t)$  ( $L$  is “evaluation at the point  $t$ ”). Such functionals are not only linear but have the special property that  $L(fg) = L(f)L(g)$ . Prove that if  $L$  is any linear functional on  $\mathcal{P}$  such that  $L(fg) = L(f)L(g)$  for all polynomials  $f$  and  $g$ , then either  $L = 0$  or there is a  $c$  in  $\mathbb{R}$  such that  $L(f) = f(c)$  for all  $f$ .

184. Let  $\mathcal{M}$  denote the vector space of real  $n \times n$  matrices and let  $\ell$  be a linear functional on  $\mathcal{M}$ . Write  $C$  for the matrix whose  $ij$  entry is  $(1/\sqrt{2})^{i+j}$ . If  $\ell(AB) = \ell(BA)$  for all  $A, B \in \mathcal{M}$ , and  $\ell(C) = 1$ , compute  $\ell(I)$ .

185. Let  $b \neq 0$ . Find the eigenvalues, eigenvectors, and determinant of  $A := \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{pmatrix}$ .

186. Let  $b, c \neq 0$ . Find the eigenvalues, eigenvectors, and determinant of  $A := \begin{pmatrix} a & b & b & \cdots & b \\ c & a & 0 & \cdots & 0 \\ c & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & 0 & 0 & \cdots & a \end{pmatrix}$ .

187. a) Let  $L : V \rightarrow V$  be a linear map on the vector space  $V$ . If  $L$  is nilpotent, so  $L^k = 0$  for some integer  $k$ , show that the map  $M := I - L$  is invertible by finding an explicit formula for  $(I - L)^{-1}$ .
- b) Apply the above result to find a particular solution of  $y' - y = 5x^2 - 3$ . [HINT: Let  $V$  be the space of quadratic polynomials and  $L := d/dx$ ].
- c) Similarly, find a particular solution of  $y'' + y = 1 - x^2$ .

188. [WINE AND WATER] You are given two containers, the first containing one liter of liquid  $A$  and the second one liter of liquid  $B$ . You also have a cup which has a capacity of  $r$  liters, where  $0 < r < 1$ . You fill the cup from the first container and transfer the content to the second container, stirring thoroughly afterwards.

Next dip the cup in the second container and transfer  $k$  liters of liquid back to the first container. This operation is repeated again and again. Prove that as the number of iterations  $n$  of the operation tends to infinity, the concentrations of  $A$  and  $B$  in both containers tend to equal each other. [Rephrase this in mathematical terms and proceed from there].

Say you now have three containers  $A$ ,  $B$ , and  $C$ , each containing one liter of different liquids. You transfer one cup from  $A$  to  $B$ , stir, then one cup from  $B$  to  $C$ , stir, then one cup from  $C$  to  $A$ , stir, etc. What are the long-term concentrations?

189. Snow White distributed 21 liters of milk among the seven dwarfs. The first dwarf then distributed the contents of his pail evenly to the pails of other six dwarfs. Then the second did the same, and so on. After the seventh dwarf distributed the contents of his pail evenly to the other six dwarfs, it was found that each dwarf had exactly as much milk in his pail as at the start.

What was the initial distribution of the milk?

Generalize to  $N$  dwarfs.

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