A Tridiagonal Matrix

We investigate the simple $n \times n$ real tridiagonal matrix:

$$M = \begin{pmatrix} \alpha & \beta & 0 & \dots & 0 & 0 & 0 \\ \beta & \alpha & \beta & \dots & 0 & 0 & 0 \\ 0 & \beta & \alpha & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & \alpha & \beta & 0 \\ 0 & 0 & 0 & \dots & \beta & \alpha & \beta \\ 0 & 0 & 0 & \dots & 0 & \beta & \alpha \end{pmatrix} = \alpha I + \beta \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} = \alpha I + \beta T,$$

where T is defined by the preceding formula. This matrix arises in many applications, such as n coupled harmonic oscillators and solving the Laplace equation numerically. Clearly M and T have the same eigenvectors and their respective eigenvalues are related by $\mu = \alpha + \beta\lambda$. Thus, to understand M it is sufficient to work with the simpler matrix T.

Eigenvalues and Eigenvectors of T

Usually one first finds the eigenvalues and then the eigenvectors of a matrix. For T, it is a bit simpler first to find the eigenvectors. Let λ be an eigenvalue (necessarily real) and $V = (v_1, v_2, \dots, v_n)$ be a corresponding eigenvector. It will be convenient to write $\lambda = 2c$. Then

$$0 = (T - \lambda I)V = \begin{pmatrix} -2c & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -2c & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2c & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2c & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -2c & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -2c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \\ v_n \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} -2cv_1 + v_2 \\ v_1 - 2cv_2 + v_3 \\ \vdots \\ v_{k-1} - 2cv_k + v_{k+1} \\ \vdots \\ v_{n-2} - 2cv_{n-1} + v_n \\ v_{n-1} - 2cv_n \end{pmatrix}$$

Except for the first and last equation, these have the form

$$v_{k-1} - 2cv_k + v_{k+1} = 0. \tag{2}$$

We can also bring the first and last equations into this same form by introducing new artificial variables v_0 and v_{n+1}, setting their values as zero: $v_0 = 0$, $v_{n+1} = 0$.

The result (2) is a *second order linear difference equation with constant coefficients* along with the *boundary conditions* $v_0 = 0$, and $v_{n+1} = 0$. As usual for such equations one seeks a solution with the form $v_k = r^k$. Equation (2) then gives $1 - 2cr + r^2 = 0$ whose roots are

$$r_\pm = c \pm \sqrt{c^2 - 1}$$

Note also

$$2c = r + r^{-1} \qquad \text{and} \qquad r_+ r_- = 1. \tag{3}$$

Case 1: $c \neq \pm 1$. In this case the two roots $r_\pm$ are distinct. Let $r := r_+ = c + \sqrt{c^2 - 1}$. Since $r_- = c - \sqrt{c^2 - 1} = 1/r$, we deduce that the general solution of (1) is

$$v_k = Ar^k + Br^{-k}, \qquad k = 2, \ldots, n-1 \tag{4}$$

for some constants A and B which.

The first boundary condition, $v_0 = 0$, gives $A + B = 0$, so

$$v_k = A(r^k - r^{-k}), \qquad k = 1, \ldots, n-1. \tag{5}$$

Since for a non-trivial solution we need $A \neq 0$, the second boundary condition, $v_{n+1} = 0$, implies

$$r^{n+1} - r^{-(n+1)} = 0, \qquad \text{so} \qquad r^{2(n+1)} = 1.$$

In particular, $|r| = 1$. Using (3), this gives $2|c| \leq |r| + |r|^{-1} = 2$. Thus $|c| \leq 1$. In fact, $|c| < 1$ because we are assuming that $c \neq \pm 1$.

Case 2: $c = \pm 1$. Then $r = c$ and the general solution of (1) is now

$$v_k = (A + Bk)c^k.$$

The boundary condition $v_0 = 0$ implies that $A = 0$. The other boundary condition then gives $0 = v_{n+1} = B(n+1)c^{n+1}$. This is satisfied only in the trivial case $B = 0$. Consequently the equations (1) have no non-trivial solution for $c = \pm 1$.

It remains to rewrite our results in a simpler way. We are in Case 1 so $|r| = 1$. Thus $r = e^{i\theta}$, $c = \cos\theta$, and $1 = r^{2(n+1)} = e^{2i(n+1)\theta}$. Consequently $2(n+1)\theta = 2k\pi$ for some $1 \leq k \leq n$ (we exclude $k = 0$ and $k = n+1$ because we know that $c \neq \pm 1$, so $r \neq \pm 1$). Normalizing the eigenvectors V by the choice $A = 1/2i$, we summarize as follows:

Theorem 1 *The $n \times n$ matrix T has the eigenvalues*

$$\lambda_k = 2c = 2\cos\theta = 2\cos\tfrac{k\pi}{n+1}, \qquad 1 \le k \le n$$

and corresponding eigenvectors

$$V_k = (\sin\tfrac{k\pi}{n+1}, \sin\tfrac{2k\pi}{n+1}, \ldots, \sin\tfrac{nk\pi}{n+1}).$$

REMARK 1. If $n = 2k+1$ is odd, then the middle eigenvalue is zero because $(k+1)\pi/(n+1) = (k+1)\pi/2(k+1) = \pi/2$.

REMARK 2. Since $2ab = a^2+b^2-(a-b)^2 \le a^2+b^2$ with equality only if $a=b$, we see that for any $x \in \mathbb{R}^n$

$$\langle x, Tx\rangle = 2(x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n) \le x_1^2 + 2(x_2^2 + \cdots + x_{n-1}^2) + x_n^2 \le 2\|x\|^2$$

with equality only if $x=0$. Similarly $\langle x, Tx\rangle \ge -2\|x\|^2$. Thus, the eigenvalues of T are in the interval $-2 < \lambda < 2$. Although we obtained more precise information above, it is useful to observe that we could have deduced this so easily.

REMARK 3. *Gershgorin's circle theorem* is also a simple way to get information about the eigenvalues of a square (complex) matrix $A = (a_{ij})$. Let D_i be the disk whose center is at a_{ii} and radius is $R_i = \sum_{j\ne i}|a_{ij}|$, so

$$|\lambda - a_{jj}| \le R_j.$$

These are the *Gershgorin disks*.

Theorem 2 (Gershgorin) *Each eigenvalues of A lies in at least one of these Gershgorin discs.*

Proof: Say $Ax = \lambda x$ and say $|x_i| = \max_j |x_j|$. The i^{th} component of $Ax = \lambda x$ is

$$(\lambda - a_{ii})x_i = \sum_{j\ne i} a_{ij}x_j$$

so

$$|(\lambda - a_{ii})x_i| \le \sum_{j\ne i}|a_{ij}||x_j| \le R_i|x_i|.$$

That is, $|\lambda - a_{ii}| \le R_i$, as claimed.

By Gershgorin's theorem, we observed immediately that all of the eigenvalues of T satisfy $|\lambda| \le 2$.

Determinant of $T - \lambda I$

We use recursion on n, the size of the $n \times n$ matrix T. It will be convenient to build on (1) and let $D_n = \det(T - \lambda I)$. As before, let $\lambda = 2c$. Then, expanding by minors using the first column of (1) we obtain the formula

$$D_n = -2cD_{n-1} - D_{n-2} \qquad n = 3, 4, \ldots. \tag{6}$$

Since $D_1 = -2c$ and $D_2 = 4c^2 - 1$, we can use (6) to define $D_0 := 1$. The relation (6) is, except for the sign of c, is identical to (2). The solution for $c \neq \pm 1$ is thus

$$D_k = As^k + Bs^{-k}, \qquad k = 0, 1, \ldots, \tag{7}$$

where

$$-2c = s + s^{-1} \qquad \text{and} \qquad s = -c + \sqrt{c^2 - 1}. \tag{8}$$

This time we determine the constants A, B from the *initial conditions* $D_0 = 1$ and $D_1 = -2c$. The result is

$$D_k = \begin{cases} \dfrac{1}{2\sqrt{c^2-1}}(s^{k+1} - s^{-(k+1)}) & \text{if} \quad c \neq \pm 1, \\ (-c)^k(k+1) & \text{if} \quad c = \pm 1. \end{cases} \tag{9}$$

For many purposes it is useful to rewrite this.

Case 1: $|c| < 1$. Then $s = -c + i\sqrt{1 - c^2}$ has $|s| = 1$ so $s = e^{i\alpha}$ and $c = -\cos\alpha$ for some $0 < \alpha < \pi$. Therefore from (9),

$$D_k = \frac{\sin(k+1)\alpha}{\sin\alpha}. \tag{10}$$

Case 2: $c > 1$. Write $c = \cosh\beta$ for some $\beta > 0$. Since $-e^{\beta} - e^{-\beta} = -2c = s + s^{-1}$, write $s = -e^{\beta}$. Then from (9),

$$D_k = (-1)^k \frac{\sinh(k+1)\beta}{\sinh\beta}, \tag{11}$$

where we chose the sign in $\sqrt{c^2 - 1} = -\sinh\beta$ so that $D_0 = 1$.

Case 3: $c < -1$. Write $c = -\cosh\beta$ for some $\beta > 0$. Since $e^t + e^{-t} = -2c = s + s^{-1}$, write $s = e^{\beta}$. Then from (9),

$$D_k = \frac{\sinh(k+1)\beta}{\sinh\beta}, \tag{12}$$

where we chose the sign in $\sqrt{c^2 - 1} = +\sinh t$ so that $D_0 = 1$.

Note that as $t \to 0$ in (10)–(12), that is, as $c \to \pm 1$. these formulas agree with the case $c = \pm 1$ in (9).

A Small Generalization

This procedure can be extended to the slightly more general tridiagonal matrices having different elements in the upper-left and lower-right entries

$$M = \begin{pmatrix} \gamma & \beta & 0 & 0 & \dots & 0 & 0 \\ \beta & \alpha & \beta & 0 & \dots & 0 & 0 \\ 0 & \beta & \alpha & \beta & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha & \beta & 0 \\ 0 & 0 & 0 & \dots & \beta & \alpha & \beta \\ 0 & 0 & 0 & \dots & 0 & \beta & \delta \end{pmatrix} = \alpha I + \beta \begin{pmatrix} a & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & b \end{pmatrix} = \alpha I + \beta T,$$

where $a = (\gamma - \alpha)/\beta$ and $b = (\delta - \alpha)/\beta$. These arises in various applications.

As in equation (1) above, we need to find the eigenvalues $\lambda = 2c$ and eigenvectors v of T, so

$$0 = (T - \lambda I)v = \begin{pmatrix} a-2c & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -2c & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2c & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2c & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -2c & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & b-2c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \\ v_n \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} (a-2c)v_1 + v_2 \\ v_1 - 2cv_2 + v_3 \\ \vdots \\ v_{k-1} - 2cv_k + v_{k+1} \\ \vdots \\ v_{n-2} - 2cv_{n-1} + v_n \\ v_{n-1} + (b-2c)v_n \end{pmatrix}.$$

Here and throughout our discussion a and b will be *real* parameters.

As before, we introduce new variables v_0 and v_{n+1} to bring the first and last equations into the same form $v_{k-1} - 2cv_k + v_{k+1} = 0$. Now $(a-2c)v_1 + v_2 = v_0 - 2cv_1 + v_2$ if we impose the boundary condition $v_0 = av_1$. Similarly, the last equation in (13) has the standard form $v_{n-1} - 2cv_n + v_{n+1} = 0$ if we add the second boundary condition $v_{n+1} = bv_n$.

Case 1: $c \neq \pm 1$. The general solution of $v_{k-1} - 2cv_k + v_{k+1} = 0$ is still

$$v_k = Ar^k + Br^{-k}, \qquad k = 2, \dots, n-1 \tag{14}$$

for some constants A and B. Here $\lambda/2 = c$ and r are related by the equations (3). Thus, r is complex if and only if $|c| < 1$. We will pick A and B so that after using the boundary conditions, we get a non-trivial solution.
Case 2: $c = \pm 1$. The general solution of $v_{k-1} - 2cv_k + v_{k+1} = 0$ is

$$v_k = (A + Bk)c^k, \qquad k = 2, \ldots, n-1 \tag{15}$$

for some constants A and B. Again, we will pick A and B so that after using the boundary conditions, we get a non-trivial solution.

Since it is simpler, we first take up CASE 2.
Case 2: $c = \pm 1$ **(details)**. The first boundary condition, $v_0 = av_1$, implies that $A(1 - ac) - Bac = 0$. Similarly, the second boundary condition, $v_{n+1} = bv_n$ gives $A(c-b) + B[(n+1)c - nb] = 0$. These equations for A and B have a non-trivial solution only if the determinant condition

$$(1 - ac)[(n+1)c - nb] + ac(c - b) = 0$$

on the constants a and b is satisfied. We rewrite this explicitly for $c = 1$ and $c = -1$:

$$(n+1) + (n-1)ab = n(a+b) \qquad \text{for} \quad c = 1 \tag{16}$$
$$(n+1) + (n-1)ab = -n(a+b) \qquad \text{for} \quad c = -1 \tag{17}$$

Case 1: $c \neq \pm 1$ **(details).** The first boundary condition gives

$$v_0 = av_1, \qquad \text{that is,} \qquad A(1 - ar) + B(1 - a/r) = 0,$$

while the second gives

$$v_{n+1} = bv_n, \qquad \text{that is,} \qquad Ar^n(r - b) + Br^{-n}(r^{-1} - b) = 0.$$

This system of two equations for A and B has a non-trivial solution if and only if the determinant of the coefficient matrix is zero. Thus r must satisfy

$$r^{-n}(1 - ar)(1/r - b) - r^n(1 - a/r)(r - b) = 0. \tag{18}$$

This can be written as a polynomial of degree $2n+2$.

$$p(r) := r^{2n+2} - \mathcal{A}r^{2n+1} + \mathcal{B}r^{2n} - \mathcal{B}r^2 + \mathcal{A}r - 1 = 0, \tag{19}$$

where $\mathcal{A} := a + b$ and $\mathcal{B} = ab$. However, two trivial roots are $r = \pm 1$, that is $c = \pm 1$, which are the roots we have excluded in this CASE 1. In Fact 2 below we show how to correlate the remaining $2n$ roots with the n eigenvalues of T.

The simplest possibilities are i). $a = b = 0$ (done above), ii). $a = 0, b = \pm 1$, iii). $a = \pm 1$, $b = 0$, iv). $a = b = \pm 1$, and v). $a = -b = \pm 1$. Although not as simple and as these, one can completely analyze the general case. In the following application to coupled oscillators, this general case corresponds to requiring a variety of boundary conditions. For instance the simplest case $a = 0$ corresponds to coupled oscillators whose left end is fixed while $a = 1$ corresponds to a free end.

We'll present some general facts and then some special examples where it is possible to give more detailed computations.

Fact 1 By Gershgorin's theorem T has one eigenvalue that is in each of the unit circles centered at $\lambda = a$ and $\lambda = b$, while the remaining $n-2$ eigenvalues are in circles centered at the origin with radius 2. Since $\lambda = 2c$ this means that there are $n-2$ values of c with $|c| < 1$ (recall that in this CASE 1 we have excluded $c = \pm 1$).

Fact 2 If we let $F(r) := r^n (r - a/r)(r - b)$, then the condition (18) asserts that $F(r) = F(1/r)$. Thus, in addition to the observation that $r = \pm 1$ is a solution, we see that if r is a solution, then so is $1/r$. If $|r| = 1$ these are complex conjugate pairs.

This helps us understand the relation between the $2n$ non-trivial roots of (18) and the n eigenvalues $\lambda = 2c$ of T. The relation follows immediately from $c = \frac{1}{2}(r + r^{-1})$ since it uses the *pair* of roots r and $1/r$. From $r_{\pm} = c \pm \sqrt{c^2 - 1}$ (from (3)) and that $n-2$ values of c satisfy $|c| < 1$ (Fact 1), we deduce that $2(n-2)$ *of the roots of* (18) *lie on the unit circle* $|r| = 1$ *as conjugate pairs*.

Fact 3 Since our matrix is symmetric, all of its eigenvalues — and hence the values c — are real. This if $r = e^{s+it}$ is a complex root of (18), then $c = \frac{1}{2}(r + r^{-1})$ must be real. But

$$r + r^{-1} = e^{s+it} + e^{-s-it} = (e^s + e^{-s}) \cos t + i(e^s - e^{-s}) \sin t.$$

Consequently, $(e^s - e^{-s}) \sin t = 0$ so either $s = 0$ or $t = k\pi$, k an integer. In other words, *the non-real roots of* (18) *lie on the unit circle* $|r| = 1$.

Fact 4 Rewrite (18) as

$$f(r) := r^{2n} \frac{(r-a)(r-b)}{(1-ar)(1-br)} = 1. \tag{20}$$

Notice that for real a and complex z, if $|a| < 1$, if $|z| < 1$ then $|(z-a)/(1-az)| < 1$ while if $|z| > 1$ then $|(z-a)/(1-az)| > 1$. This implies that if $|a| < 1$ and $|b| < 1$, then for $|r| < 1$ we have $|f(r)| < 1$ while if $|r| > 1$ then $|f(r)| > 1$. It therefore cannot be satisfied for either $|r| < 1$ or $|r| > 1$. Thus the only possibility is $|r| = 1$. Checking the cases $a = \pm 1$ and $b = \pm 1$ separately, we find that *for all real* $|a| \le 1$ *and* $|b| \le 1$, *all the roots of* (18) *lie on the unit circle* $|r| = 1$.

We can also deduce this conclusion from Fact 1 since if $|a| < 1$ and $|b| < 1$, then all of the eigenvalues of T lie in the disk centered at the origin, radius 2; consequently $|c| \le 1$ which implies all the roots of (18) lie on the unit circle $|r| = 1$.

Special Examples:

$a = b = 0$.
This is the example we did at the beginning of this note. Equation (18) is then $r^{2n+3} = 1$. Excluding the trivial roots $r = \pm 1$ corresponding to $c = \pm 1$ we recover the same results.

$a = b = 1$ and $a = b = -1$.
If $a = b = 1$ then (18) becomes

$$0 = r^{2n}(r-1)^2 - (1-r)^2 = (r-1)^2(r^{2n}-1)$$

so $r = 1$ or $r^{2n} = 1$ etc. Write $r = e^{i\theta}$. Then $\theta = k\pi/n$, $k = 1, \ldots, n-1$ (as usual, we exclude $r = \pm 1$, that is, $k = 0$ and $k = n$). Consequently,

$$\lambda = 2c = 2\cos k\pi/n, \qquad \text{for} \qquad k = 1, \ldots, n-1.$$

The case $a = b = -1$ is essentially identical.

$ab = 1$.
This extends the previous example. Here (18) becomes

$$0 = r^{2n}[r^2 - (a+1/a)r + 1] - [1 - (a+1/a)r + r^2] = (r-a)(r-1/a)(r^{2n}-1)$$

whose roots are $r = a$, $r = 1/a$ and the roots of $r^{2n} = 1$ — except $r = \pm 1$.

$a = 0$, $b = \pm 1$ and $a = \pm 1$, $b = 0$.
If $a = 0$ and $b = 1$, then (18) becomes

$$0 = r^{2n+1}(r-1) - (1-r) = (r-1)(r^{2n+1}+1),$$

so $r = 1$ or $r^{2n+1} = -1$ etc. The other possibilities are essentially identical.

$a = \pm 1$, $b = \mp 1$.
In both of these cases (18) becomes

$$0 = r^{2n}(r^2-1) - (1-r^2) = (r^2-1)(r^{2n}+1),$$

so $r = \pm 1$ or $r^{2n} = -1$ etc.

$a = -b$.
Here (19) becomes

$$0 = p(r) = r^{2n}(r^2 - a^2) - (1 - a^2r^2). \tag{21}$$

We claim that if $a^2 < (n+1)/(n-1)$ *all* the roots of $p(r) = 0$ lie on the unit circle $|r| = 1$, while if $a^2 > (n+1)/(n-1)$, then $p(r) = 0$ has four real roots: $r = r_1 > 1$, the paired root $r = 1/r_1 < 1$ as well as $r = -r_1$ and $r = -1/r_1$, the remainder lying on the unit circle. In other words, if $a^2 < (n+1)/(n-1)$, then all the eigenvalues of T satisfy $|\lambda| < 2$ while if $a^2 > (n+1)/(n-1)$, then are exactly two real eigenvalues with $|\lambda| > 2$. All of the eigenvalues appear in pairs: λ and $-\lambda$.

To verify these assertions, since (21) only involves r^2, it is sufficient to find the positive zeroes of $g(t) := t^n((t-q)-(1-qt)$, where $q = a^2 \geq 0$. Clearly $g(0) = -1$, $g(1) = 0$, $g(t) > 0$ for large positive t and $g(1/t) = -t^{-(n-1)}g(t)$. Also $g'(t) = (n+1)t^n - nqt^{n-1} + q$ and $g''(t) = nt^{n-2}[(n+1)t-(n-1)q]$. Thus $g''(t) = 0$ at $t = 0$ and $t = t_0 := \frac{n-1}{n+1}q$ so $g'(t_0) = [1-t_0^{n-1}]q$. This shows that if $t_0 \leq 1$, that is, if $q \leq \frac{n+1}{n-1}$, then $g'(t) \geq 0$ for all $t \geq 0$ and hence $g(t)$ has only only one positive real zero, located at $t = 1$. On the other hand, if $t_0 > 1$, then $g(t)$ has three real positive zeroes: at $t_1 = 1$, $t = t_2$ for some $t_2 > 1$, and $t = t_3 = 1/t_2$.

$a = 0$.
Here (19) becomes

$$0 = p(r) = r^{2n+1}(r-b) - (1-br) = r^{2n+2} - br^{2n+1} + br - 1. \tag{22}$$

Since $a = 0$ we already know that p has $2(n-1)$ zeroes on $|r| = 1$ in addition to the two trivial zeroes $r = \pm 1$. Thus there are at most two additional real zeroes: $r = r_1$ and $r = 1/r_1$. We claim that these real zeroes exist (and are not $+1$ or -1) if and only if $|b| > \frac{n+1}{n}$; otherwise all of the non-trivial zeroes are on the unit circle $|r| = 1$.

The reasoning is similar to the previous example. We use $p'(r) = (2n+2)r^{2n+1} - (2n+1)r^{2n} + b$ and $p''(r) = 2(2n+1)r^{2n-1}[(n+1)r - nb]$. Then $p(0) = -1$, $p(\pm 1) = 0$, $p'(0) = b$ and the only zero of p'' is at $r_0 := \frac{nb}{n+1}$. Now $p'(r_0) = b(1-r_0^{2n})$. Say $b > 0$. If $r_0 \leq 1$, that is, if $b \leq \frac{n+1}{n}$, then $p'(r) \geq 0$ for all $r \geq 0$ so $p(r)$ has only the trivial zero at $r = 1$. The reasoning for $r < 0$ and $b < 0$ are similar.

$n = 2$.
The characteristic polynomial of T is $\lambda^2 - (a+b)\lambda + ab - 1$ so its eigenvalues are

$$\lambda_{\pm} = \frac{a+b}{2} \pm \sqrt{\left(\frac{a-b}{2}\right)^2 + 1}.$$

In particular, if $|a-b| >> 1$ then $\lambda_{\pm} \approx a, b$ while if $a \approx b$, then $\lambda_{\pm} \approx \frac{1}{2}(a+b) \pm 1$.

General Case

One can use the reasoning of the examples $a = -b$ and $a = 0$ ang obtain a fairly complete discussion of the general case. The key quantities are $p'(\pm 1)$, so we record these here:

$$p'(1) = 2[(n+1) - n\mathcal{A} + (n-1)\mathcal{B}], \qquad p'(-1) = -2[(n+1) + n\mathcal{A} + (n-1)\mathcal{B}]$$

From (19) and the above facts we know that $p(0) = -1$, $p(\pm 1) = 0$, $p(\pm\infty) > 0$, and that there are at most four other real roots, each appearing as a pair $r, 1/r$. Becaues of this pairing, the roots $r = \pm 1$ can only have multiciplities 1, 3, and 5, so if $p'(1) = 0$, then also $p''(1) = 0$, etc. These imply

1, If $\mathbf{p'(1) < 0}$: then p has exactly two non-trivial positive zeroes, at some $0 < r_1 < 1$ and $r_2 := 1/r_1 > 1$.

2. If $\mathbf{p'(-1) < 0}$: then p has exactly two non-trivial negative zeroes, at some $-1 < r_1 < 0$ and $r_2 := 1/r_1 < -1$.

3. If $\mathbf{p'(1) > 0}$: then p has either four or no real positive roots.

4. If $\mathbf{p'(-1) < 0}$: then p has either four or no real non-trivial negative roots.

5. If $\mathbf{p'(1) = 0}$ and $\mathbf{p'''(1) < 0}$: then p has exactly five positive zeroes: $r = +1$ with multiciplity three and two non-trivial positive zeroes, at some $0 < r_1 < 1$ and $r_2 := 1/r_1 > 1$. If $\mathbf{p'(1) = 0}$ and $\mathbf{p'''(1) > 0}$, then the only positive roots of p are $r = 1$ with multiciplity three. Finally,, if $\mathbf{p'(1) = p'''(1) = 0}$, then also $p''(1) = p''''(1) = 0$ so $r = 1$ is a root with multiciplity five and there can not be any other non-trivial real roots.

6. If $\mathbf{p'(-1) = 0}$ and $\mathbf{p'''(1) > 0}$: then p has exactly five negative zeroes: $r = -1$ with multiciplity three and two non-trivial negative zeroes, at some $-1 < r_1 < 0$ and $r_2 := 1/r_1 < -1$. If $\mathbf{p'(-1) = 0}$ and $\mathbf{p'''(1) < 0}$, then the only negative roots of p are $r = -1$ with multiciplity three. Finally, if $\mathbf{p'(-1) = p'''(-1) = 0}$, then $r = -1$ is a root with multiciplity five and there can not be any other non-trivial real roots.

We'll prove only the first of these assertions; the others are similar. Thus we assume $\mathbf{p'(1) < 0}$. Since $p(1) = 0$, then $p(r) < 0$ for small $r > 1$. Thus, if p had two real roots greater than 1, it would also have a third. Consequently it would have six non-trivial positive roots, which is impossible,

CONSEQUENCE: If a and b have opposite sign (or if one of them is zero), then p can have at most one pair of non-tivial positive roots and one pair of non-trivial negative roots. This eliminates the only ambiguity in 3) and 4) above so one obtains a complete analysis. The examples $a = -b$ and $a = 0$ are both included here.

If a and b have the same sign, then the only missing piece is the ambiguity in 3) and 4) above.

A Vibrating String (coupled oscillators)

Say we have n particles with the same mass m equally spaced on a string having tension τ. Let y_k denote the vertical displacement if the kth mass. Assume the ends of the string are fixed; this is the same as having additional particles at the ends, but with zero displacement: $y_0 = 0$ and $y_{n+1} = 0$. Let ϕ_k be the angle the segment of the string between the kth and k+1st particle makes with the horizontal. Then Newton's second law of motion applied to the kth mass asserts that

$$m\ddot{y}_k = \tau \sin\phi_k - \tau \sin\phi_{k-1}, \qquad k = 1, \ldots, n. \tag{23}$$

If the particles have horizontal separation h, then $\tan\phi_k = (y_{k+1} - y_k)/h$. For the case of small vibrations we assume that $\phi_k \approx 0$; then $\sin\phi_k \approx \tan\phi_k = (y_{k+1} - y_k)/h$ so we can rewrite (23) as

$$\ddot{y}_k = p^2(y_{k+1} - 2y_k + y_{k-1}), \qquad k = 1, \dots, n, \tag{24}$$

where $p^2 = \tau/mh$. This is a system of second order linear constant coefficient differential equations with the boundary conditions $y_0(t) = 0$ and $y_{n+1}(t) = 0$. As usual, one seeks special solutions of the form $y_k(t) = v_k e^{\alpha t}$. Substituting this into (24) we find

$$\alpha^2 v_k = p^2(v_{k+1} - 2v_k + v_{k-1}), \qquad k = 1, \dots, n,$$

that is, α^2 is an eigenvalue of $p^2(T - 2I)$. From the work above we conclude that

$$\alpha_k^2 = -2p^2(1 - \cos\tfrac{k\pi}{n+1}) = -4p^2 \sin^2 \tfrac{k\pi}{2(n+1)}, \qquad k = 1, \dots, n,$$

so

$$\alpha_k = 2ip \sin\tfrac{k\pi}{2(n+1)}, \qquad k = 1, \dots, n.$$

The corresponding eigenvectors V_k are the same as for T. Thus the special solutions are

$$Y_k(t) = V_k e^{2ipt \sin\frac{k\pi}{2(n+1)}}, \qquad k = 1, \dots, n,$$

where $Y(t) = (y_1(t), \dots, y_n(t))$.