Math 508 — December 4, 2008

Exam 2

Jerry L. Kazdan — 10:30 – 11:50

Directions This exam has two parts, Part A has 10 True-False problems (30 points, 3 points each). Part B has 5 traditional problems (70 points, 14 points each).
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: True/False (answer only, no reasons). 10 problems, 3 points each).
Circle T or **F** in in each problem.

1. **T F** A bounded sequence $\{a_n\}$ of real numbers always has a convergent subsequence.

2. **T F** A series $\sum_{n=1}^{\infty} a_n$ of complex numbers converges if and only if the corresponding sequence of partial sums is bounded.

3. **T F** A closed and bounded subset of a complete metric space must be compact.

4. **T F** If A and B are compact subsets of a metric space, then $A \cup B$ is also compact.

5. **T F** If M is any metric space and $f : M \to \mathbb{R}$ is any continuous real-valued function, then the function $g : M \to \mathbb{R}$ defined by $g(x) := (f(x))^2$ is always continuous.

6. **T F** If $f : X \to Y$ is a continuous map between metric spaces, and $f(X)$ is compact, then X is compact.

7. **T F** A compact subset of a metric space is always complete.

8. **T F** Let $\{x_n\}$ be a sequence of points in a metric space. If two subsequences of this sequence converge, then they must converge to the same number.

9. **T F** If $f : [0, 1] \to \mathbb{R}$ is a continuous function and $\int_0^1 f(x)\,dx = 0$, then $f(x)$ is positive somewhere and negative somewhere in this interval (unless it is identically zero).

10. **T F** $f(x) := \displaystyle\sum_{1}^{\infty} \frac{\sin(3^n \pi x)}{2^n}$ is a continuous function on $\mathbb{R}$.

Part B: Traditional Problems (5 problems, 14 points each)

B–1. Let $f : [-2, 2]$ be a smooth function with the property that

$$f(-1) = 1, \qquad f(0) = 0, \qquad f(1) = 2.$$

Show that at some point $c \in (-1, 1)$ we have $f''(c) > 0$. In fact, find an explicit constant $m > 0$ so that $f''(c) \geq m$.

B–2. Let $A(t)$ and $B(t)$ be $n \times n$ matrices that are differentiable for $t \in [a, b]$ and let $t_0 \in (a, b)$. Directly from the *definition* of the derivative, show that the product $M(t) := A(t)B(t)$ is differentiable at $t = t_0$ and obtain the usual formula for $M'(t_0)$.

B–3. Let $w(x)$ be a smooth function that satisfies $w'' - c(x)w = 0$, where $c(x) > 0$ is a given function.

a) Show that w cannot have a local positive maximum (that is, a maximum at an interior point where the function is positive). Also show that w cannot have a local negative minimum.

b) [Uniqueness] If you also know that $w(0) = a$ and $w(1) = b$, prove that there is *at most* one solution $w(x) \in C^2([0, 1])$ with these properties.

B–4. Let $f(x)$ and $K(x, y)$ be a given continuous real valued functions for $x, y \in [0, 2]$, and, say $|K(x, y)| \leq M$. Show that if $0 < a \leq 2$ is sufficntly small, the integral equation

$$u(x) = f(x) + \int_0^x K(x, y)\, u(y)\, dy$$

has a unique continuous solution $u(x)$ for $x \in [0, a]$.

B–5. Let $\varphi_n(t)$ be a sequence of smooth real-valued functions with the properties

$$(a)\ \varphi_n(t) \geq 0, \qquad (b)\ \varphi_n(t) = 0 \text{ for } |t| \geq 1/n, \qquad (c)\ \int_{-\infty}^{\infty} \varphi_n(t)\, dt = 1.$$

Note: because of (b), this integral is only over $-1/n \leq t \leq 1/n$.

Assume $f(x)$ is uniformly continuous for all $x \in \mathbb{R}$ and define

$$f_n(x) := \int_{-\infty}^{\infty} f(x - t)\varphi_n(t)\, dt.$$

Show that $f_n(x)$ converges uniformly to $f(x)$ for all $x \in \mathbb{R}$. [SUGGESTION: Use $f(x) = f(x)\left(\int_{-\infty}^{\infty} \varphi_n(t)\, dt\right) = \int_{-\infty}^{\infty} f(x)\varphi_n(t)\, dt$. Also, note *explicitly* where you use the uniform continuity of f].

REMARK: One can show that the approximations f_n are also smooth. Thus, this proves that you can approximate a continuous function *uniformly* on any compact set by a smooth function.