Math 508 **Exam 2** Jerry L. Kazdan
December 4, 2008 10:30 – 11:50

DIRECTIONS This exam has two parts, Part A has 10 True-False problems (30 points, 3 points each). Part B has 5 traditional problems (70 points, 14 points each).
Closed book, no calculators or computers– but you may use one $3'' \times 5''$ card with notes on both sides.

Part A: True/False (answer only, no reasons). 10 problems, 3 points each).
Circle T or **F** in in each problem.

1. **T F** A bounded sequence $\{a_n\}$ of real numbers always has a convergent subsequence.

   True Rudin, p. 51 Theorem 3.6b)

2. **T F** A series $\sum_{n=1}^{\infty} a_n$ of complex numbers converges if and only if the corresponding sequence of partial sums is bounded.

   False EXAMPLE: $\sum(-1)^n$.

3. **T F** A closed and bounded subset of a complete metric space must be compact.

   False EXAMPLE: The unit sphere in ℓ_2.

4. **T F** If A and B are compact subsets of a metric space, then $A \cup B$ is also compact.

   True Rudin, p.38 2.35 Corollary

5. **T F** If M is any metric space and $f : M \to \mathbb{R}$ is any continuous real-valued function, then the function $g : M \to \mathbb{R}$ defined by $g(x) := f(x)^2$ is always continuous.

   True Given $x_0 \in M$ and $\epsilon > 0$, pick $\delta > 0$ so that if $d(x, x_0) < \delta$, then $|f(x) - f(x_0)| < \epsilon$. Let $M = |f(x_0)|$. Then

   $$|g(x) - g(x_0)| = |f(x)^2 - f(x_0)^2| = |\big(f(x) - f(x_0)\big)\big(f(x) - f(x_0) + 2f(x_0)\big)| \le \epsilon(\epsilon + 2M).$$

6. **T F** If $f : X \to Y$ is a continuous map between metric spaces, and $f(X)$ is compact, then X is compact.

   False EXAMPLE: $X = \mathbb{R}$, $f(x) = 0$ for all $x \in \mathbb{R}$.

7. **T F** A compact subset of a metric space is always complete.

   True Rudin, p. 54 (after Definition 3.12)

8. **T F** Let $\{x_n\}$ be a sequence of points in a metric space. If two subsequences of this sequence converge, then they must converge to the same number.

False EXAMPLE: In $\mathbb{R}$, $\{(-1)^n\}$.

9. **T F** If $f : [0, 1] \to \mathbb{R}$ is a continuous function and $\int_0^1 f(x)\,dx = 0$, then $f(x)$ is positive somewhere and negative somewhere in this interval (unless it is identically zero).

True If f is not identically zero, then it is either positive somewhere or negative somewhere (or both). Say it is positive at x_0. Then by continuity, it is positive in a neignborhood of x_0. If $f(x) \ge 0$, everywhere, then $\int_0^1 f(x)\,dx > 0$, a contradiction. Thus f must be negative at some x_1 – and hence also in a neighborhood of x_1.

10. **T F** $f(x) := \sum_1^\infty \frac{\sin(3^n \pi x)}{2^n}$ is a continuous function on $\mathbb{R}$.

True Since $\left|\frac{\sin(3^n \pi x)}{2^n}\right| \le \frac{1}{2^n}$, by the Weierstrass M-Test the series converges uniformly and absolutely – and hence to a continuous function.

Part B: Traditional Problems (5 problems, 14 points each)

B–1. Let $f : [-2, 2]$ be a smooth function with the property that

$$f(-1) = 1, \qquad f(0) = 0, \qquad f(1) = 2.$$

Show that at some point $c \in (-1, 1)$ we have $f''(c) > 0$. In fact, find an explicit constant $m > 0$ so that $f''(c) \ge m$.

Solution By the mean value theorem applied to the intervals $[-1, 0]$ and $[0, 1]$ there are points $a \in (-1, 0)$ and $b \in (0, 1)$ so that $f'(a) = -1$ and $f'(b) = 2$. Applying the mean value theorem to f', we conclude there is a point $c \in (a, b)$ such that $f''(c) = 3/(b-a) > 3/2$.

B–2. Let $A(t)$ and $B(t)$ be $n \times n$ matrices that are differentiable for $t \in [a, b]$ and let $t_0 \in (a, b)$. Directly from the *definition* of the derivative, show that the product $M(t) := A(t)B(t)$ is differentiable at $t = t_0$ and obtain the usual formula for $M'(t_0)$.

Solution [Caution: Usually $A(t)$ and $B(t)$ will *not* commute.]

$$\begin{aligned}\frac{M(t_0+h) - M(t_0)}{h} &= \frac{A(t_0+h)B(t_0+h) - A(t_0)B(t_0)}{h}\\ &= \frac{[A(t_0+h) - A(t_0)]}{h}B(t_0+h) + A(t_0)\frac{[B(t_0+h) - B(t_0)]}{h}.\end{aligned}$$

Since both A and B are differentiable at t_0, in the limit as $h \to 0$ we find that M is differentiable at t_0 and

$$M'(t_0) = A'(t_0)B(t_0) + A(t_0)B'(t_0).$$

B–3. Let $w(x)$ be a smooth function that satisfies $w'' - c(x)w = 0$, where $c(x) > 0$ is a given function.

a) Show that w cannot have a local positive maximum (that is, a maximum at an interior point where the function is positive). Also show that w cannot have a local negative minimum.

Solution If w has a positive maximum at an interior point x_0, then $w''(x_0) \le 0$ and $w(x_0) > 0$. Since $c > 0$, this gives $w''(x_0) - c(x_0)w(x_0) < 0$, which is a contradiction. If $w(x)$ has a local negative minimum, then the function $-w(x)$ has a local positive maximum, which we have just shown cannot occur.

b) [Uniqueness] If you also know that $w(0) = a$ and $w(1) = b$, prove that there is *at most* one solution $w(x) \in C^2([0, 1])$ with these properties.

Solution Say there are two functions $u(x)$ and $v(x)$ that satisfy $w'' - c(x)w = 0$ and have the same boundary conditions. Let $z(x) := u(x) - v(x)$. Then $z'' - cz = 0$ with $z(0) = 0$ and $z(1) = 0$. If $z(x)$ is not identically zero, then it has either a positive maximum or a negative minimum at an interior point. But by part a), this cannot happen. Thus $z(x) = 0$ everywhere in $[0, 1]$. Consequently $u(x) = v(x)$ in this interval.

B–4. Let $f(x)$ and $K(x, y)$ be a given continuous real valued functions for $x, y \in [0, 2]$, and, say $|K(x, y)| \le M$. Show that if $0 < a \le 2$ is sufficntly small, the integral equation

$$u(x) = f(x) + \int_0^x K(x, y)\, u(y)\, dy$$

has a unique continuous solution $u(x)$ for $x \in [0, a]$.

Solution We use the Principle of Contracting Mapping. For our complete metric space we use $C([0, a])$ (with the uniform norm) for some $0 < a \le 2$ which will be specified below. Let

$$T\varphi(x) := f(x) + \int_0^x K(x, y)\, \varphi(y)\, dy$$

and seek a fixed point u of T.

Clearly for all $0 < a \le 2$ we have $T : C([0, a])$ to $C([0, a])$. We need only show that T is contracting for some $a \le 2$. But

$$T\varphi(x) - T\psi(x) = \int_0^x K(x, y)\, [\varphi(y) - \psi(y)]\, dy$$

so

$$|T\varphi(x) - T\psi(x)| \le Ma\|\varphi - \psi\|.$$

Pick $a < 1/M$. Since the right side of the above inequality is independent of $x \in [0, a]$, then with $c = Ma$,

$$\|T\varphi - T\psi\| \le c\|\varphi - \psi\|,$$

so T is contracting. Thus the original integral equation has a unique solution.

Remark. By modifying this reasoning we can even use $a = 2$. One approach to do this is to show that some power of T is contracting on $[0, 2]$.

B–5. Let $\varphi_n(t)$ be a sequence of smooth real-valued functions with the properties

$$(a)\ \varphi_n(t) \geq 0, \qquad (b)\ \varphi_n(t) = 0 \text{ for } |t| \geq 1/n, \qquad (c)\ \int_{-\infty}^{\infty} \varphi_n(t)\,dt = 1.$$

Note: because of (b), this integral is only over $-1/n \leq t \leq 1/n$.

Assume $f(x)$ is uniformly continuous for all $x \in \mathbb{R}$ and define

$$f_n(x) := \int_{-\infty}^{\infty} f(x-t)\varphi_n(t)\,dt.$$

Show that $f_n(x)$ converges uniformly to $f(x)$ for all $x \in \mathbb{R}$. [SUGGESTION: Use $f(x) = f(x)\left(\int_{-\infty}^{\infty} \varphi_n(t)\,dt\right) = \int_{-\infty}^{\infty} f(x)\varphi_n(t)\,dt$. Also, note *explicitly* where you use the uniform continuity of f].

REMARK: One can show that the approximations f_n are also smooth. Thus, this proves that you can approximate a continuous function *uniformly* on any compact set by a smooth function.

Solution Using the suggestion,

$$f_n(x) - f(x) = \int_{-1/n}^{1/n} [f(x-t) - f(x)]\,\varphi_n(t)\,dt.$$

Since f is uniformly continuous, given any $\epsilon > 0$ there is a $\delta > 0$ so that $|f(y) - f(x)| < \epsilon$ for *any* x, y that satisfy $|y - x| < \delta$. Pick some N with $1/N < \delta$. Then for any $n \geq N$, if $|t| \leq 1/n$ then $|(x-t) - x| \leq 1/n < \delta$ so $|f(x-t) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$. Consequently

$$|f_n(x) - f(x)| \leq \epsilon \int_{|t| \leq 1/n} \varphi_n(t)\,dt = \epsilon.$$

Because the right side is independent of x, we have $\|f_n - f\| \leq \epsilon$ in the uniform norm.