Two Inequalities for Integrals of Vector Valued Functions

Theorem Let $F : [a,b] \to \mathbb{R}^n$ be a continuous vector-valued function. Then

$$\left\| \int_a^b F(t)\,dt \right\| \le \int_a^b \|F(t)\|\,dt$$

with equality if and only if there is a continuous scalar valued function $\varphi(t) \ge 0$ such that $F(t) = \varphi(t)V$ where $V := \int_a^b F(t)\,dt$.

Proof: We begin with the observation that for any vectors X and $V \neq 0$, the proof of the Schwarz inequality shows that $\langle X, V\rangle \le \|X\|\|V\|$ with equality if and only if $X = cV$ for some constant $c \ge 0$. Thus if V is a constant vector, then for any t

$$\langle F(t), V\rangle \le \|F(t)\|\|V\|$$

with equality if and only if $F(t) = \varphi(t)V$ for some scalar valued function $\varphi(t) \ge 0$.
Thus for any V

$$\langle \int_a^b F(t)\,dt, V\rangle = \int_a^b \langle F(t), V\rangle \le \int_a^b \|F(t)\|\|V\|\,dt,$$

with equality if and only if $F(t) = \varphi(t)V$ for some continuous scalar valued function $\varphi(t) \ge 0$. To complete the proof we choose $V := \int_a^b F(t)\,dt$ so the left side of the above inequality becomes $\|V\|^2$ and then cancel $\|V\|$ from both sides (unless V=0 in which case the theorem is trivial).

Corollary [Mean Value Inequality] Let $\gamma : [a,b] \to \mathbb{R}^n$ define a curve whose first deriative is continuous. Then

$$\|\gamma(b) - \gamma(a)\| \le \int_a^b \|\gamma'(t)\|\,dt,$$

with equality if and only if $\gamma'(t) = \varphi(t)[\gamma(b) - \gamma(a)]$ for some continuous scalar valued function $\varphi(t) \ge 0$ (so the velocity vector is along the straight line from $\gamma(a)$ to $\gamma(b)$).
Since $\int_a^b \|\gamma'(t)\|\,dt$ can be interpreted as the *arc length* of the curve for $a \le t \le b$, this inequality has a natural geometric interpretation.

Proof: By the Fundamental Theorem of Calculus

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)\,dt.$$

Now apply the above theorem.