Math508, Fall 2008 **Jerry L. Kazdan**

Problem Set 0: Rust Remover

DUE: Thurs. Sept. 4, 2008. These problems will not be collected.

You should already have the techniques to do these problems. They should not take much time.

1. a) Graph the points (x,y) in the plane $\mathbb{R}^2$ that satisfy $|y-x| > 2$.

   b) Graph the points $z = x+iy$ in the complex plane that satisfy $1 < |z-i| < 2$.

2. a) If $r(\neq 0)$ is a rational number and x is irrational, show that both $r+x$ and rx are *irrational*.

   b) Prove that there is no rational number whose square is 12.

   c) Write the complex number $z = \dfrac{1}{a+ib}$ in the form $c+id$, where a, b, c are d are real numbers. Of course assume $a+ib \neq 0$.

3. a) Show that for any positive integer n, the number $2^{n+2}+3^{2n+1}$ is divisible by 7.

   b) Does this use that fact that we customarily write our integers base 10?.

   c) Generalize part (a).

4. Let z, w, $z_1, \ldots, z_n$ be complex numbers

   a) Prove the *triangle inequality*: $|z_1+\cdots+z_n| \leq |z_1|+\cdots+|z_n|$ (first do the case $n=2$).

   b) Show that $||z|-|w|| \leq |z-w|$.

5. Let the continuous function $f(\theta)$, $0 \leq \theta \leq 2\pi$ represent the temperature along the equator at a certain moment, say measured from the longitude at Greenwich.. Show there are antipodal points with the *same* temperature.

6. A certain function $f(x)$ has the property that $\int_0^x f(t)\,dt = e^x \cos x + C$. Find both f and the constant C.

7. If $b \geq 0$, show that for every real c the equation $x^5+bx+c=0$ has exactly one real root.

8. Let $p(x) := x^3 + cx + d$, where c, and d are real. Under what conditions on c and d does this has three distinct real roots? [HINT: Sketch a graph of this cubic. Observe that if there are three distinct real roots then there is a local maximum and the polynomial is positive there. What about a local min?].

9. Prove that the function $\sin x$ is not a polynomial. That is, there is no polynomial

$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$

with real coefficients so that $\sin x = p(x)$ for all real numbers x. In your proof you can use any of the standard properties of the function $\sin x$.

[Last revised: August 28, 2008]