Problem Set 1

DUE: Thurs. Sept. 11, 2008. *Late papers will be accepted until 1:00 PM Friday*.

Many of these problems are from the Rudin text.

1. (Rudin, p. 23 #19) Suppose $a \in \mathbb{R}^k$, $b \in \mathbb{R}^k$, and $x \in \mathbb{R}^k$. Find all $c \in \mathbb{R}^k$ and $r > 0$ (depending on a and b) such that $|x-a| = 2|x-b|$ is satisfied if and only if $|x-c| = r$. [ANSWER: $3c = 4b-a$, $3r = 2|b-a|$].

   As an alternate, you may prefer the following generalization. For real $\lambda > 0$, $\lambda \neq 1$, consider the points $x \in \mathbb{R}^k$ that satisfy
   $$|x-a| = \lambda|x-b|.$$
   Show that these points lie on a sphere. Part of this is to find the center and radius of this sphere in terms of a, b and λ. What if $\lambda = 1$?

2. (Rudin, p.43 #2) A complex number is *algebraic* if it is a root of a polynomial $a_0 z^n + \cdots + a_n$ whose coeffients are all integers. Prove that the set of all algebraic numbers is countable. [HINT: For every positive integer N there are only finitely many equations with $n + |a_0| + \cdots + |a_n| = N$.]

3. (Rudin, P.43 #3) Prove there exist real numbers that are not algebraic.

4. (Rudin, P.43 #4) Is the set of all irrational real numbers countable?

5. (Rudin, p. 22 #6) The point of this problem is, for any real $b > 1$ and any real x to define b^x. So far we can only compute b^x for the special cases $x = n$ and for $x = 1/n$, where $n = 1, 2, \ldots$. First we extend this to rational x and then to all real x.

   Fix $b > 1$. Let m, n, p, q be integers with $n > 0$, $q > 0$. Set $r = m/n = p/q$.

   a) Prove that $(b^m)^{1/n} = (b^p)^{1/q}$. Thus, it makes sense to define $b^r = (b^m)^{1/n}$.

   b) If r and s are rational, prove that $b^{r+s} = b^r b^s$.

   c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that for r rational
   $$b^r = \sup B(r).$$
   Hence it makes sense to *define* $b^x = \sup B(x)$ for all real x.

   d) With this definition, prove that for all real x, y: $b^{x+y} = b^x b^y$.

6. (Rudin, p. 23 #17) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with the Euclidean distance, prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2.$$

Interpret this geometrically as a statement about parallelograms.

[Last revised: September 8, 2008]