Homework Set 10 (Due: Never)

1. An $n \times n$ matrix $R : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal matrix* if it preserves the length of each vector $x \in \mathbb{R}^n$, that is,
$$\|Rx\| = \|x\| \qquad \text{for all} \qquad x \in \mathbb{R}^n.$$

   a) If R and S are both $n \times n$ orthogonal matrices, show that their product, SR is also an orthogonal matrix [thus they form a *group*].

   b) Show that the set of $n \times n$ orthogonal matrices form a compact set in $\mathbb{R}^{n^2}$.

2. Show that the function $F(x) := \sum_{k=1}^{\infty} \dfrac{\cos(k^2 x)}{k^2 + x^2}$ is continuous for all real x.

3. Let $f(z) = \sum_0^{\infty} a_n z^n$ and say the power series has radius of convergence R. Let $r < R$.

   a) Show that the series for the formal derivative $f'(z) = \sum_0^{\infty} n a_n z^{n-1}$ converges uniformly in the disc $\{|z| \le r\}$.

   b) Conclude that the function $f \in C^{\infty}(|z| < 1)$.

4. Let $f \in C([0,\infty])$ be a continuous function with the property that $\lim_{x\to\infty} f(x) = c$. Show that
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x)\,dx = c.$$

5. Let $K(x,y)$ be a continuous function of x and y for x and y in the interval $[0,1]$ and let
$$h(x) = \int_0^1 K(x,y)\,dy.$$
Show that $h(x)$ is a continuous function of x for $x \in [0,1]$.

6. a) If $f(x)$ and $K(x,y)$ are continuous function of x and y for x and y in the interval $0 \le x \le 1$, show that if $|\lambda|$ is sufficiently small, then the integral equation
$$u(x) = f(x) + \lambda \int_0^1 K(x,y) u(y)\,dy$$
has a unique solution $u \in C([0,1])$.

b) Investigate the integral equation

$$u(x) = f(x) + \lambda \int_0^1 e^{x-y} u(y)\, dy$$

and find all $\lambda \in \mathbb{R}$ for which this has a unique solution for any $f \in C([0,1])$. The result shows the smallness requirement on λ in part a) cannot be removed completely. [This is short, explicit, and elementary – but perhaps not obvious. Write $\int_0^1 e^{x-y} u(y)\, dy = e^x \int_0^1 e^{-y} u(y)\, dy$].