Homework Set 3

DUE: Tues. Sept. 30, 2008. Late papers accepted until 1:00 Wednesday.

1. (Rudin, p.44 #20) Are the closures and interiors of connected sets always connected? [Look at subsets of $\mathbb{R}^2$.]

2. Let K be a compact set in a metric space $\mathcal{M}$ and let $p \in \mathcal{M}$ be a point *not* in K. Define the distance $d(p,K)$ between p and K as

$$d(p,K) = \inf_{x\in K} d(p,x).$$

   a) Show there is at least one point $q \in K$ that has this minimum distance, so $d(p,q) = d(p,K)$
   b) Is there a *unique* such point q? Proof or counterexample.
   c) Is the assertion in part a) still true if you only assume that K is a closed (but not compact) subset of $\mathbb{R}^2$? Proof or counterexample.

3. a) Calculate $\lim_{n\to\infty} \dfrac{5n+17}{n+2}$.
   b) Let $a_n := \dfrac{3n^2-2n+17}{n^2+21n+2}$. Calculate $\lim_{n\to\infty} a_n$.

4. (Rudin, p.78 #2) Calculate $\lim_{n\to\infty} \sqrt{n^2+n}-n$.

5. If $c > 0$, show that $\dfrac{c^n}{n!} \to 0$ as $n \to \infty$.

6. If $b > 1$ and $s \in \mathbb{R}$, show that $\dfrac{n^s}{b^n} \to 0$ as $n \to \infty$.

7. (Rudin, p.78 #3)Let $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2+\sqrt{s_n}}$, $n = 1,2,3,\ldots$. Prove that $\{s_n\}$ converges to some number s and that $s_n < 2$.

8. (Rudin, p. 78 #5) Let $\{a_n\}$ and $\{b_n\}$ be any real bounded sequences.

a) Show that

$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided the sum on the right is not of the form $\infty - \infty$.

b) Give an explicit example where strict inequality can occur.

9. Let $p_k = (x_k, y_k) \in \mathbb{R}^2$, $k = 1, 2, \dots$ be a sequence of points in the plane (with the usual Euclidean metric). Show that $\{p_k\}$ converges to $p = (x, y)$ if and only if $x_k \to x$ and $y_k \to y$.

10. [NEWTON] Let $A > 0$ and $x_1 > 0$. Define $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{A}{x_n}\right)$. The following steps show that $x_n \to \sqrt{A}$.

a) Show that after the first term, the sequence $\{x_n\}$ is monotonically decreasing and that $x_2^2 \ge A$ (hence $x_n^2 \ge A$ for $n \ge 2$).

b) Show the x_n converge to some real number L and, using the definition of x_n, that $L^2 = A$.

11. Given a real sequence $\{a_k\}$, let $C_n = \dfrac{a_1 + \cdots + a_n}{n}$ be the sequence of averages (*arithmetic mean*).

a) Give an example where the a_n's doesn't converge but the averages do converge.

b) If the averages converge, must the a_n's be bounded? (Proof or counterexample)

c) If a_k converges to A, show that also C_n converges, and to A.

d) If $b_k \in \mathbb{R}$ are positive and $b_k \to B$, show that their *geometric mean* also converge to B, that is $[b_1 b_2 \cdots b_n]^{1/n} \to B$.

12. If $\{b_k\}$ is a sequence of positive numbers, prove the arithmetic-geometric mean inequality

$$[b_1 b_2 \cdots b_n]^{1/n} \le \frac{b_1 + \cdots + b_n}{n}.$$

When does equality hold?

[Last revised: September 30, 2008]