Math 508, Fall 2008 **Jerry Kazdan**

Homework Set 7

DUE: Thurs. Nov. 6, 2008. Late papers accepted until 1:00 Friday.

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. Which of the following are uniformly continuous in the set $\{x \ge 0\}$? Justify your assertions.

   a). $f(x) = 2 + 3x$ b). $g(x) = \sin 2x$ c). $h(x) = x^2$ d). $k(x) = \sqrt{x}$,

2. Let a smooth function $g(x)$ have the three properties: $g(0) = 2$ $g(1) = 0$ $g(4) = 6$. Show that at some point $0 < c < 4$ one has $g''(c) > 0$. Better yet, find a number $m > 0$ so that $g''(c) \ge m > 0$.

   Is it true that g'' must be positive at at least one point $0 < c < 1$? Proof or counterexample.

3. a) Show that $\sin x$ is not a polynomial.

   b) Show that $\sin x$ is not a rational function, that is, it cannot be the quotient of two polynomials.

   c) Let $f(t)$ be periodic with period 1, so $f(t+1) = f(t)$ for all real t. If f is not a constant, show that it cannot be a rational function. that is, f cannot be the quotient of two polynomials.

   d) Show that e^x is not a rational function.

4. a) If a smooth function $f(x)$ has the property that $f''(x) \ge 0$ for all x, show that it is *convex*, that is, at every point the graph of the curve $y = f(x)$ lies above all its tangent lines.

   b) Let $v(x)$ be a smooth real-valued function for $0 \le x \le 1$. If $v(0) = v(1) = 0$ and $v''(x) > 0$ for all $0 \le x \le 1$, show that $v(x) \le 0$ for all $0 \le x \le 1$.

   c) Prove that the function e^x is convex.

   d) Show that $e^x \ge 1 + x$ for all real x.

5. a) Let $p(x) := x^3 + cx + d$, where c, and d are real. Under what conditions on c and d does this has three distinct real roots? [ANSWER: $c < 0$ and $d^2 < -4c^3/27$].

   b) Generalize to the real polynomial $p(x) := ax^3 + bx^2 + cx + d$ $(a \ne 0)$ by a change of variable $t = x - \alpha$ (with a clever choice of α) to reduce to the above special case.

6. If $h : \mathbb{R} \to \mathbb{R}$ is a differentiable function that satisfies $h'(t) \le ch(t)$, where c is a constant, show that $h(t) \le e^{ct} h(0)$ for all $t \ge 0$.

7. Say $u(t)$ satisfies $u'' + b(t)u' + c(t)u = 0$, where $b(t)$ and $c(t)$ are bounded functions. Let $E(t) := \frac{1}{2}(u'^2 + u^2)$.

a) Show that $E'(t) \leq \gamma E(t)$, where γ is a constant. [SUGGESTION: Use the inequality $|2xy| \leq x^2 + y^2$].

b) Use the result of the previous problem to deduce that if $u(0) = 0$ and $u'(0) = 0$, then $u(t) = 0$ for all t.

8. Let $w(x)$ be a smooth function that satisfies $w'' - c(x)w = 0$, where $c(x) > 0$ is a given function, show that w cannot have a local positive maximum (that is, a local maximum where the function is positive). Also show that w cannot have a local negative minimum.

9. a) For any integer $n \geq 0$, show that $\lim_{x \searrow 0} \dfrac{e^{-1/x}}{x^n} = 0$.

   b) Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}.$$
   Sketch the graph of f.

   c) Show that f is a smooth function for all real x.

   d) Show that each of the following are smooth and sketch their graphs:
$$g(x) = f(x)f(1-x) \qquad h(x) = \frac{f(x)}{f(x) + f(1-x)}$$
$$k(x) = h(x)h(4-x) \qquad K(x) = k(x+2),$$
$$\varphi(x,y) = K(x)K(y),\ (x,y) \in \mathbb{R}^2 \qquad \Phi(x) = K(\|x\|),\ x = (x_1, x_2) \in \mathbb{R}^2$$

10. [Interpolation] Let $x_0 < x_1 < x_2$ be distinct real numbers and $f(x)$ a smooth function.

   a) Show there is a unique quadratic polynomial $p(x)$ with the property that $p(x_j) = f(x_j)$ for $j = 0, 1, 2$.

   b) [Remainder term in interpolation] If b is in the open interval (x_0, x_2) with $b \neq x_j$, $j = 0, 1, 2$, show there is a point c (depending on b) in the interval (x_0, x_2) so that
$$f(b) = p(b) + \frac{f'''(c)}{3!}(b - x_0)(b - x_1)(b - x_2).$$
   This estimate is related to the procedure used to find the remainder in a Taylor polynomial. [SUGGESTION: Define the constant M by
$$f(b) = p(b) + M(b - x_0)(b - x_1)(b - x_2),$$
   and look at
$$g(x) := f(x) - [p(x) + M(x - x_0)(x - x_1)(x - x_2)].]$$