Homework Set 8

DUE: Thurs. Nov. 13, 2008. Late papers accepted until 1:00 Friday.

Note: We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. Let $\mathbf{r}(t)$ define a smooth curve that does not pass through the origin.

   a) If the point $\mathbf{a} = \mathbf{r}(t_0)$ is a point on the curve that is closest to the origin (and *not* an end point of the curve), show that the position vector $\mathbf{r}(t_0)$ is perpendicular to the tangent vector $\mathbf{r}'(t_0)$.

   b) What can you say about a point $\mathbf{b} = \mathbf{r}(t_1)$ that is *furthest* from the origin?

2. Consider two smooth plane curves $\gamma_1, \gamma_2 \colon (0,1) \to \mathbb{R}^2$ that do not intersect. Suppose P_1 and P_2 are interior points on γ_1 and γ_2, respectively, such that the distance $|P_1P_2|$ is minimal. Prove that the straight line P_1P_2 is perpendicular to *both* curves.

3. Say $\gamma(t) : \mathbb{R} \to \mathbb{R}^2$ defines a smooth curve in the plane.

   a) If $\gamma(0) = 0$ and $\|\gamma'(t)\| \le c$, show that for any $T \ge 0$, $\|\gamma(T)\| \le cT$. Moreover, show that equality can occur if and only if one has $\gamma(t) = vt$ where v is a vector that does not depend on t.

4. Let $\gamma(t) = (x(t), y(t))$ define a curve in the plane parametrized by t. We say the parametrization is *smooth* if the functions $x(t)$, $y(t)$ are smooth *and* the velocity, $\gamma'(t) \ne 0$ [see part (b) below for why we require that $\gamma'(t) \ne 0$].

   a) Find a smooth parameterization of the curve $y = x^2$.

   b) The curve $y = |x|$ for $-1 \le x \le 1$ is clearly not smooth at $x = 0$ – but it has a parameterization $\gamma(t) = (x(t), y(t))$ with both $x(t)$ and $y(t)$ smooth everywhere. However $\gamma'(t)$ will be zero at the corner [intuition: if $\gamma(t)$ describes the position of a particle, have the partcle slow down to a stop near corners].

   Use the function defined in Homework Set 7, Problem 9(b) to find smooth functions $x(t)$, $y(t)$ that parameterize $y = |x|$ for $-1 \le x \le 1$.

5. [Derivative of the Determinant] Let $A(t)$ be a family of square matrices whose elements depend smoothly on a real parameter t.

   a) If $A(0) = I$, show that

   $$\left.\frac{d \det A(t)}{dt}\right|_{t=0} = \operatorname{trace} A'(0).$$

   [The *trace* of a matrix is the sum of the elements on the diagonal].

   b) If $A(t)$ is invertible, compute the derivative of the determinant of $A(t)$ at $t = t_0$. [SUGGESTION: Let $B(t) = A(t)A^{-1}(t_0)$ and use the previous part.]

6. Since the function $f(x) := x^2$ is monotonic, by what we showed in class, it is Riemann integrable. Using a partition with each interval having equal width, directly compute

$$\int_0^a x^2\,dx.$$

In case it helps, $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.