Homework Set 9

DUE: Tues. Nov. 25, 2008. Late papers accepted until 1:00 Wednesday.

1. Find a continous function f and a constant C so that $\displaystyle\int_0^x f(t)\,dt = x\cos x + 8e^x + C$.

2. Let $f:[0,\,1]\to\mathbb{R}$ be a continuous function.
   a) If $\int_0^1 f(x)\,dx = 0$, prove that $f(x)$ is positive somewhere and negative somewhere in this interval (unless it is identically zero).
   b) If $f:[0,\,1]\to\mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* continuous functions g prove that $f(x)=0$ for all $x\in[0,\,1]$.
   c) If $f:[0,\,1]\to\mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* C^1 functions g that satisfy $g(0)=g(1)=0$, must it be true that $f(x)=0$ for all $x\in[0,\,1]$? Proof or counterexample.

3. [HÖLDER'S INEQUALITY] Let $p,q>1$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$.
   a) Show that $st \le \dfrac{s^p}{p}+\dfrac{t^q}{q}$ for all $s,t>0$.
   [SUGGESTION: There are many ways to prove this. One is to show that for any $a>0$ and $s\ge 0$ the maximum of $h(s):=as-s^p/p$ occurs at $s=a^{1/(p-1)}$.]
   b) Use this to show that for any complex numbers a_k, b_k
   $$\sum_{k=1}^n |a_k b_k| \le \left[\sum_{k=1}^n |a_k|^p\right]^{1/p}\left[\sum_{k=1}^n |b_k|^q\right]^{1/q}.$$
   [SUGGESTION: First do the special case $\left[\sum_{k=1}^n |a_k|^p\right]^{1/p}=1$ and $\left[\sum_{k=1}^n |b_k|^q\right]^{1/q}=1$. Then reduce the general case to this special case.]
   If $p=q=1/2$ this is the Schwarz inequality.
   c) Similarly, show that for any continuous functions f, g
   $$\int_a^b |f(x)g(x)|\,dx \le \left[\int_a^b |f(x)|^p\,dx\right]^{1/p}\left[\int_a^b |g(x)|^q\,dx\right]^{1/q}.$$

4. Let $p,q>1$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$. and let $X:=(x_1,\ldots,x_n)\in\mathbb{R}^n$ and $f\in\mathbb{C}([a,b])$. Use Hölder's inequality (above) to prove the triangle inequality for the norms
$$\|X\|_p := \left[\sum_{k=1}^n |x_k|^p\right]^{1/p} \quad\text{and}\quad \|f\|_p := \left[\int_a^b |f(x)|^p\,dx\right]^{1/p}.$$

5. Continuing the notation of the previous problem, define the norms

$$\|X\|_\infty ;= \max_k \{|x_k|\} \quad \text{and} \quad \|f\|_\infty := \max_{x\in[a,b]} |f(x)|.$$

a) Show that $\lim_{p\to\infty} \|X\|_p = \|X\|_\infty$.
b) Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$

6. Compute $\displaystyle\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x)|\, dx$.

7. Let $f(x)$ be a continuous function for $0 \le x \le 1$. Evaluate $\displaystyle\lim_{n\to\infty} n \int_0^1 f(x)x^n\, dx$. (Justify your assertions.)

8. For $x > 0$ define the function

$$H(x) = \int_1^x \frac{1}{t}\, dt.$$

Since the integrand, $1/t$ is a continuous function on the interval $[1,x]$ (if $x \ge 1$) or $[x.1]$ (if $x \le 1$), this is Riemann integrable.

Use the definition of the Riemann integral directly to show that for any $y > 0$,

$$H(x) + H(y) = H(xy), \tag{1}$$

thus establishing that $H(x)$ has the basic property of the logarithm.
SUGGESTION: First prove (1) assuming $x \ge 1$ (and any $y > 0$) by rewriting (1) in the form $H(x) = H(xy) - H(y)$, that is,

$$\int_1^x \frac{1}{t}\, dt = \int_y^{xy} \frac{1}{s}\, ds$$

and use a geometric argument that relates a Riemann sum for the integral on the left to a corresponding Riemann sum on the right.. [First try the special case $x = 2$, $y = 2$.]

If $0 < x < 1$, then $1/x > 1$, so the result (1) follows from the case $x \ge 1$ by the clever chain:

$$\begin{aligned} H(x) + H(y) &= H(x) + H(\frac{1}{x}xy) = H(x) + [H(\frac{1}{x}) + H(xy)] \\ &= H(\frac{1}{x}) + H(x) + H(xy) = H(\frac{1}{x}x) + H(xy) = H(1) + H(xy) = H(xy). \end{aligned}$$

9. Let $p(x)$ be a real polynomial of degree n. The following uses the inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)\, dx$.

a) If p is orthogonal to the constants, show that p has at least one real zero in the interval $\{0 < x < 1\}$.

b) If p is orthogonal to all polynomials of degree at most one, show that p has at least two distinct real zeros in the interval $\{0 < x < 1\}$.

c) If p is orthogonal to all polynomials of degree at most $n-1$, show that p has exactly n distinct real zeros in the interval $\{0 < x < 1\}$.

BONUS PROBLEMS

These are more challenging. If you do any of these, please give your solutions directly to me by Tuesday, Dec. 2.

Bonus Problem 1 Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function.

a) Show that $\lim_{\lambda\to\infty} \int_0^1 f(x)\sin(\lambda x)\,dx = 0$.

b) (generalization) If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous with period P, show that

$$\lim_{\lambda\to\infty} \int_0^1 f(x)\varphi(\lambda x)\,dx = \overline{\varphi} \int_0^1 f(x)\,dx,$$

where $\overline{\varphi} := \frac{1}{P}\int_0^P \varphi(t)\,dt$ is the average of φ over one period.

Bonus Problem 2 Let $\mathcal{C}$ be the ring of continuous functions on the interval $0 \le x \le 1$.

a) If $0 \le c \le 1$, show that the subset $\{\, f \in \mathcal{C} \mid f(c) = 0 \,\}$ is a maximal ideal.

b) Show that every maximal ideal in $\mathcal{C}$ has this form. [Caution: This is false for the ring of continuous functions on the open interval $0 < x < 1$.]