# Advanced Analysis: Outline

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This outline of the course is to help you step back and get a larger view of what we have done. Since this is only an outline, I will often not explicitly state the precise assumptions needed for the assertion to hold.

## Important Sets

$\mathbb{R}$: THE REAL NUMBERS Primary features: they form a *field*, are *ordered*, and are *complete*. The completeness is the primary feature that distinguishes from the much smaller field $\mathbb{Q}$ of rational numbers. The function $|x|$ is a *norm* that measures the size of a real number.

Basic concepts: *convergence* of a sequence, *limit point* of a sequence and of a set, *open*, *closed*, and *bounded* sets, *Cauchy sequences*, *countable* and *uncountable* sets. Two important theores are the Bolzano-Weierstrass Theorem (bounded sequences have convergent subsequences) and the Heine-Borel Theorem (when open covers have a finite sub-cover). These theorems are associated with the *compactness* of a set.

$\mathbb{C}$: COMPLEX NUMBERS: $z = x + iy$ They also form a complete field but are not ordered. The norm $|z| := \sqrt{x^2 + y^2}$ measures the size. The triangle inequality $|z + w| \le |z| + |w|$ is basic. The concepts of convergence and limit point etc. extend immediately. The convergence of an *infinite series* arises frequently. We define $e^z$, $\cos z$, and $\sin z$ using power series. Euler's beautiful observation $e^{ix} = \cos x + i \sin x$ is valuable.

GENERALIZATION: $\mathbb{R}^k$, $\mathbb{C}^k$, $\ell_1$, $\ell_2$, the set of $n \times k$ real or complex matrices. These are important *normed linear spaces*. The norm $\|X\|$ is used to define a *metric* $d(X,Y) := \|X - Y\|$.

The concepts of convergence and limit points generalize immediately, as do those of open and closed sets, Cauchy sequence, compactness, etc. If $A$ is a square matrix we define $e^A$ using the power series.

The concept of *connectedness* arises and is important.

These spaces are all complete (proving the completeness of $\ell_1$ and $\ell_2$ takes some work). Subsets of these, such as the closed unit ball of points $X$ where $\|X\| \leq 1$ are also metric spaces, although they are not linear spaces (if $X$ and $Y$ are in the unit ball, $X+Y$ might not be). Most of our metric spaces will simply be subsets of normed linear spaces.

The point of introducing the examples $\ell_1$ and $\ell_2$ is that while closed bounded sets are compact in $\mathbb{R}^k$ and $\mathbb{C}^k$, they are usually *not* compact in $\ell_1$ or $\ell_2$ – or in other important examples (such as the set of continuous functions on $[0,1]$ with the uniform norm) that we will meet soon.

In some normed linear spaces there is an *inner product* $\langle X, Y\rangle$ and the norm is given by $\|X\| := \sqrt{\langle X, X\rangle}$. These spaces are particularly easy to use – and arise frequently in applications. In them one can define two vectors $X$ and $Y$ to be *orthogonal* if $\langle X, Y\rangle = 0$, in which case the Pythagorean theorem holds:

$$\|X+Y\|^2 = \|X\|^2 + \|Y\|^2.$$

The spaces $\mathbb{R}^k$, $\mathbb{C}^k$, $\ell_2$, and $n \times k$ matrices with their "Euclidean" norms all have natural inner products. One can show that the norm $\|\ \|$ in a normed linear space comes from an inner product if and only if the norm satisfies the *parallelogram identity*

$$\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2$$

You can use this to show that points $X = (x_1, x_2) \in \mathbb{R}^2$ with the norm $\|X\|_1 := |x_1| + |x_2|$ does *not* come from an inner product. Similarly, the norm on $\ell_1$ does not come from an inner product.

In the study of Fourier series and quantum mechanics,the virtues of inner product spaces becomes evident,

REFERENCES The Hoffman text, Chapters 1, 2, and parts of 6.1-6.3.

## Maps $f : S \to T$ between Metric Spaces

CONTINUITY For maps $f : S \to T$ between any two sets, not necessarily in a metric space, there is the elementary concept of when the map is *one-to-one* (injective), *onto* (surjective), or both one-to-one and onto (bijective). If $f : S \to T$ and $g : T \to U$, one can also compose maps: $(g \circ f)(x) := g(f(x))$.

If there is more structure, such as in metric spaces $S$ with metric $d_S(x,y)$ and $T$ with metric $d_T(u,v)$, one can define when a map $f : S \to T$ is *continuous* at a point $x \in S$ and when it is *uniformly continuous* in the whole set $S$.

The simplest examples of continuous maps $f : \mathbb{R} \to \mathbb{R}$ are polynomials.

If $f : S \to T$ and $g : S \to T$, where $T$ is a linear space we can add maps $f(x) + g(x)$ and multiply them by constants $cf(x)$ If $T$ is the real (or complex) numbers, we can also multiply and divide maps: $f(x)g(x)$, $f(x)/g(x)$. There are standard theorems stating that if $f$ and $g$ are continuous, then so are these new maps, assuming you never attempt to divide by zero. The composition of continuous maps is continuous.

There is a fundamental theorem that states that continuous maps $f : S \to T$ between metric spaces preserve basic properties:

a) If $S$ is compact, then $f$ is uniformly continuous.

b) If $S$ is compact, then $f(S)$ is compact.

c) If $S$ is connected, then $f(S)$ is connected.

As a special case, if $S$ is compact and $f : S \to \mathbb{R}$ then there are points $x_{\text{max}}$ and $x_{\text{min}}$ where $f$ has its maximum and minimum values.

Another special case is the *intermediate value theorem* for real-valued functions.

THE DERIVATIVE: $f : \mathbb{R} \to \mathbb{R}$ There are two simple situations both of which lead to the derivative:

a) Given the graph of a curve $y = f(x)$, find the *slope* at $x_0$.

b) If $y = f(t)$ is the position of a particle at time $t$, find the *velocity*.

The standard definition is as the limit of a difference quotient:

$$\frac{df}{dx}(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Proving that the limit exists is part of the task.

One gets the usual formulas for the derivative of a sum, product, and quotient.

The *mean value theorem* is a fundamental tool in understanding the derivative. It is a simple consequence of Rolle's theorem.

Similarly one can define second derivatives and higher order derivatives for a real-valued function $f(x)$ – assuming they exist. The Taylor polynomial $p_k(x)$ of degree $k$ near $x = x_0$ gives an approximation to $f$ near $x_0$:

$$p_k(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots a_k(x - x_0)^k, \quad \text{where} \quad a_j = \frac{f^{(j)}(x_0)}{j!}.$$

The formula for the error, $f(x) - p_k(x)$, is a generalization of the mean value theorem

$$f(x) - p_k(x) = \frac{f^{(k+1)}(c(x))}{(k+1)!}(x - x_0)^{k+1},$$

where $c(x)$ is some (unknown) point between $x_0$ and $x$. In the special case where $j = 1$ this gives the second derivative test for maxima and minima.

It is standard to write $C([a,b])$ for the set of continuous real-valued functions on the interval $[a,b])$ and $C^1([a,b])$ for the functions which, with their first derivatives, are continuous on $[a,b]$. Some people write $C^0([a,b])$ instead of $C([a,b])$. We say a function is *smooth* if it has derivatives of all orders, and write the set of all such functions as $C^\infty([a,b])$.

All of these sets of functions (or *function spaces*) are linear spaces. To measure the size of a function in $C([a,b])$ and $C^1([a,b])$ we use the norms, respectively,

$$\|f\|_{C([a,b])} = \sup_{a\le x\le b} |f(x)| \tag{1}$$

$$\|f\|_{C^1([a,b])} = \sup_{a\le x\le b} |f(x)| + \sup_{a\le x\le b} |f'(x)| \tag{2}$$

Since these functions are all assumed continuous on the closed interval $[a,b]$, we could have written max instead of sup.

It will be convenient (and standard) to also use (1) as a norm on the linear space of all *bounded functions* on the interval $[a,b]$. This is then called the *uniform norm* and written

$$\|f\|_\infty = \sup_{a\le x\le b} |f(x)|. \tag{3}$$

For continuous functions, this is identical to (1). (The reason for the notation $\|f\|_\infty$ is clarified in Problem Set 9, #7b.)

THE DERIVATIVE: $f:\mathbb{R}\to\mathbb{R}^k$. For this example, it is helpful to think of $f(t)$ as a vector in $\mathbb{R}^k$ giving the position of a particle at time $t$. Then the derivative, $f'(t)$ gives the *velocity* of the particle. This vector $f'(t)$ is tangent to the path of the particle. Its length, $\|f'(t)\|$, gives the *speed*.

As an exercise it is easy to show that if the speed of a particle is constant, then the acceleration is perpendicular to the velocity. The key step is to use the inner (="dot") product to express the speed:

$$\text{const} = \|f'(t)\|^2 = \langle f'(t), f'(t)\rangle.$$

Now take the derivative.

THE RIEMANN INTEGRAL The motivation problem here is to find the area under the graph of a curve $y = f(x)$ for $a \le x \le b$. If $f(x) = \text{const}$, then the region is a rectangle whose area is known. For the general case, partition the $x$-axis into $n$ segments $a = x_0 < x_1 < \cdots x_n = b$ and think of the thin strips as approximate rectangles. Let the maximum width of the strips go to zero and hope that this approximation converges, independently of how one chose the strips. If so, the function is Riemann integrable.

The basic theorem is that any continuous real-valued function on a closed and bounded interval is Riemann integrable. The critical ingredient is that a continuous function on a compact set is uniformly continuous. One consequence is that *piecewise continuous* functions are Riemann integrable. Riemann integrable.

The integral has the following basic properties:

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \tag{4}$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \tag{5}$$

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx \tag{6}$$

$$\le \sup_{x\in[a,b]} |f(x)|(b-a). \tag{7}$$

There are other vital properties, such as *integration by parts* and *change of variable*.

If we let $J(f) = \int_a^b f(x)\,dx$, the first two properties above say that $J(cf) = cJ(f)$ and that $J(f+g) = J(f)+J(g)$. In other words, that $J$ is a *linear map*.

THE FUNDAMENTAL THEOREM OF CALCULUS This asserts that the integral is the same as the "anti-derivative". There are two versions: a). the derivative of the integral, and b). the integral of the derivative.

a) If $F(x) := \int_a^b f(t)\,dt$, then $F'(x) = f(x)$.

b) $\int_a^b f'(x)\,dx = f(b) - f(a)$

An immediate consequence is that the unique solution $u(t)$ of the simplest differential equation $du/dt = f(t)$ with initial condition $u(0) = c$ is

$$u(t) = c + \int_0^t f(s)\,ds$$

REFERENCES The Hoffman text, Chapters 3, 4.1-4.4.

## Sets of Maps (Function Spaces)

POINTWISE AND UNIFORM CONVERGENCE Just as real numbers arise as limits of sequence of rational numbers, many maps arise as limits of simpler maps. The simplest example is the exponential function $e^x$ as the limit of the polynomials $p_n(x) = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$.

The most intuitive version of convergence of a sequence of real-valued functions $f_n(x)$ to $f(x)$ on a set $S$ is if at every point $x$ of $S$ we have $f_n(x)$ converge to $f(x)$. The elementary example where $S = [0,1]$ and $f)n(x) = x^n$ shows that even though the $f_n$ are each well-behaved, their pointwise limit is discontinuous at $x = 1$.

One problem that arises frequently is, if the continuous functions $f_n(x) \to f(x)$ pointwise for $x \in [a,b]$, does

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx\ ?$$

There are many simple examples (as $f(x) = nx^n$ on $[0,1]$) where this fails. Integration involves taking a limit. So here we are interchanging two limiting processes. It would be surprising if the results would always be the same.

A simple intutive example is to let $V(r,t)$ be the volume of water in a pail at time $t$ where initially there is a hole of radius $r$ in the bottom and the pail is full of water. The limits

$$\lim_{t\to\infty}\left[\lim_{r\to 0} V(r,t)\right] \quad \text{and} \quad \lim_{r\to 0}\left[\lim_{t\to\infty} V(r,t)\right]$$

will obviously be different. In the second case you are putting the lock on the barn after the horse has escaped.

The concept of *uniform convergence* is valuable to show that under appropriate circumstances one can interchange limiting processes. Let $f_n(x)$ be a sequence of bounded functions. We use the uniform norm defined above (3) to define convergence: $f_n$ converge to $f$ *uniformly* on the interval $[a,b]$ if $\|f_n - f\|_\infty \to 0$. Clearly uniform convergence implies pointwise convergence.

One simple but fundamental result is that if $f_n(x) \in \mathbb{C}([a,b])$ converge uniformly to $f(x)$, then $f(x)$ is also continuous. A reformulation is that *the function space $C([a,b])$ with the uniform norm is a complete metric space*.

Another easy result is that if $f_n \in C([a,b])$ converge uniformly to $f$ on the finite interval $[a,b]$, then

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx\ .$$

An important ingredient in mathematics is to recycle and reuse old ideas in new situations. The concepts such an convergence and compactness that we investigated for sets of real numbers and now needed for sequences of maps.

[Last revised: December 5, 2010]