Contracting Maps and an Application

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One often effective way to show that an equation $g(x) = b$ has a solution is to reduce the problem to find a *fixed point* x of a *contracting map* T, so $Tx = x$. For instance, assume V is a linear space and $g : V \to V$. Define a new map $T : V \to V$ by $T(x) := x - g(x) + b$. Then clearly x is a fixed point of T if and only if it solves $g(x) = b$.

Our setting is a metric space $\mathcal{M}$ with metric $d(x, y)$ and a map $T : \mathcal{M} \to \mathcal{M}$. We say that T is a *contracting map* if there is some c with $0 < c < 1$ such that

$$d(Tx, Ty) \le cd(x,y) \quad \text{for all} \quad x, y \in \mathcal{M} \tag{1}$$

so T contracts the distance between points. The following theorem was found by Banach who abstracted the essence of Picard's existence theorem for ordinary differential equations. We will reverse the historical order and first prove Banach's version.

Theorem 1 [PRINCIPLE OF CONTRACTING MAPS] *Let $\mathcal{M}$ be a complete metric space and $T : \mathcal{M} \to \mathcal{M}$ a contracting map. Then T has a unique fixed point $p \in \mathcal{M}$.*

PROOF The *uniqueness* is short. Say p and q are fixed points. Then by the contracting condition

$$d(p,q) = d(Tp, Tq) \le cd(p, q).$$

Since $c < 1$, the only possibility is that $d(p,q) = 0$ so $p = q$.

To prove the *existence* of a fixed point, pick any $x_0 \in \mathcal{M}$ and inductively define the successive approximations x_k by $x_k = Tx_{k-1}$ for $k = 1, 2, \ldots$. Then using (1)

$$d(x_{k+1}, x_k) = d(Tx_k, Tx_{k-1}) \le cd(x_k, x_{k-1})$$

so by induction

$$d(x_{k+1}, x_k) \le c^k d(x_1, x_0). \tag{2}$$

We use this to show that the x_k form a Cauchy sequence. Pick any $n > k$. Then by the triangle inequality and (2)

$$\begin{aligned} d(x_n, x_k) &\le d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{k+1}, x_k) \\ &\le \left(c^{n-1} + c^{n-2} + \cdots c^k\right) d(x_1, x_0) \\ &\le \frac{c^k}{1-c} d(x_1 x_0). \end{aligned}$$

Because $c < 1$ the x_k form a Cauchy sequence. The completeness of $\mathcal{M}$ implies there is a point $p \in \mathcal{M}$ with $x_k \to p$. By the continuity of T (which follows from (1)),

$$Tp = T(\lim_{k\to\infty} x_k) = \lim_{k\to\infty} Tx_k = \lim_{k\to\infty} x_{k+1} = p.$$

This proves that p is a fixed point of T.

In this proof, the x_k are better and better approximations to the fixed point p. That is why this theorem is sometimes called the *method of successive approximations*.

The following small generalization is sometimes useful. Write $T^2x = T(T(x))$, $T^3x = T(T^2(x))$, etc. and observe that $TT^k = T^{k+1} = T^kT$ (composition of maps is associative).

Corollary 2 *If some power of T, say T^k, is a contraction, then T has a unique fixed point $p \in \mathcal{M}$.*

PROOF By the theorem, there is a unique $p \in \mathcal{M}$ such that $T^kp = p$. We claim that $Tp = p$. Because $d(T^kx, T^ky) \le cd(x,y)$ for some $c < 1$, this follows from

$$d(Tp, p) = d(T(T^kp), T^kp) = d(T^k(Tp), T^kp) \le cd(Tp, p).$$

The fixed point of T is unique because any fixed point of T is also a fixed point of T^k,

Application

Let $A(t)$ be an $n \times n$ matrix and $f(t) \in \mathbb{R}^n$ a vector. Assume that the elements of A and the components of F are continuous functions of t for $|t| \le a$. We want to show that the (linear) system of ordinary differential equations

$$\frac{du}{dt} = A(t)u + f(t) \quad \text{with initial condition} \quad u(0) = c \tag{3}$$

has a unique solution. Here $c \in \mathbb{R}^n$ is a given and the vector $u(t)$ is to be found.

Theorem 3 *There is a α with $0 < \alpha \le a$ such that the initial value problem* (3) *has a unique solution $u(t)$ in $C^1([-\alpha, \alpha])$.*

PROOF Using the fundamental theorem of calculus we integrate both sides of the differential equation in (3) and observe it is enough to find a function $u(t) \in C([-\alpha, \alpha])$ that satisfies

$$u(t) = c + \int_0^t A(s)u(s)\,ds + F(t), \quad \text{where} \quad F(t) := \int_0^t f(s)\,ds. \tag{4}$$

Note that although we only seek a continuous $u(t)$ that satisfies (4), by the fundamental theorem of calculus it follows from (4) that u is indeed in $C^1([-\alpha, \alpha])$ and satisfies (3).

For any $\varphi \in C([-a,a])$ define the map $T : C([-a,a]) \to C([-a,a])$ by

$$(T\varphi)(t) := c + \int_0^t A(s)\varphi(s)\,ds + F(t). \tag{5}$$

Note that the right hand side is a continuous function of t (it is even in C^1), the new function $T\varphi$ is also continuous.

Key observation: if u is a fixed point of T, so $Tu = u$, then it will be the desired solution of (4). Thus we seek a complete metric space $\mathcal{M}$ so that $T : \mathcal{M} \to \mathcal{M}$ and is contracting. For

$\mathcal{M}$ we'll use $C([-\alpha,\alpha])$ with the uniform norm and pick $\alpha \le a$ shortly. The completeness of $\mathcal{M}$ is simply that the uniform limit of continuous functions is continuous.

Since $\alpha \le a$, we know immediately that T maps $\mathcal{M}$ to itself.

To verify the contracting condition, note that for any $\varphi(s)$ and $\psi(s)$ in $\mathbb{C}(|s| \le a)$

$$(T\varphi)(t) - (T\psi)(t) = \int_0^t A(s)[\varphi(s) - \psi(s)]\,ds. \tag{6}$$

We use that the matrix $A(s)$ is continuous for $|s| \le a$. Thus it is bounded: $|A(s)| \le m$ for all $|s| \le a$. Consequently

$$\begin{aligned} |A(s)[\varphi(s) - \psi(s)]| &\le m|\varphi(s) - \psi(s)| \\ &\le m\|\varphi - \psi\|_\infty, \end{aligned}$$

where on the right side we used the uniform norm on $C([-\alpha,\alpha])$. Thus,

$$|(T\varphi)(t) - (T\psi)(t)| \le m\alpha\|\varphi - \psi\|_\infty \tag{7}$$

for all $|t| \le \alpha$. Since the right side does not depend on t, we can take the max of the left side over all $|t| \le \alpha$ and conclude that

$$\|T\varphi - T\psi\|_\infty \le m\alpha\|\varphi - \psi\|_\infty.$$

To satisfy the contracting condition it is evident that we need only pick α so that $m\alpha < 1$, that is, $\alpha < 1/m$. This completes the proof.

To get the contracting condition, we needed to choose $\alpha < 1/m$. However by using Corollary 2, this restriction can be avoided and we can let $\alpha = a$. To do this, we need only replace the crude inequality (7) by using that we are really integrating only over $[0,t]$. (Here we'll assume $t \ge 0$; the case $t \le 0$ is identical). This gives

$$|(T\varphi)(t) - (T\psi)(t)| \le mt\|\varphi - \psi\|_\infty. \tag{8}$$

Thus

$$|(T^2\varphi)(t) - (T^2\psi)(t)| = \int_0^t A(s)[T\varphi(s) - T\psi(s)]\,ds \le m^2\|\varphi - \psi\|_\infty \int_0^t s\,ds = m^2\|\varphi - \psi\|_\infty \frac{t^2}{2}.$$

Repeating this we find

$$|(T^k\varphi)(t) - (T^k\psi)(t)| \le m^k\|\varphi - \psi\|_\infty \int_0^t \frac{s^{k-1}}{(k-1)!}\,ds = m^k\|\varphi - \psi\|_\infty \frac{t^k}{k!}.$$

Picking k so large that $m^k a^k/k! < 1$ shows that T^k is contracting on $C([0,a])$ so by Corollary 2 we are done.

[Last revised: November 21, 2014]