Problem Set 0: Rust Remover

DUE: These problems will not be collected.

You should already have the techniques to do these problems, although they may take some thinking.

1. Show that for any positive integer n, the number $2^{n+2}+3^{2n+1}$ is divisible by 7.

2. Say you have k linear algebraic equations in n variables; in matrix form we write $AX=Y$. Give a proof or counterexample for each of the following.
   a) If $n=k$ there is always *at most one* solution.
   b) If $n>k$ you can *always* solve $AX=Y$.
   c) If $n>k$ the nullspace of A has dimension greater than zero.
   d) If $n<k$ then for *some* Y there is *no* solution of $AX=Y$.
   e) If $n<k$ the *only* solution of $AX=0$ is $X=0$.

3. Let A and B be $n\times n$ matrices with $AB=0$. Give a proof or counterexample for each of the following.
   a) $BA=0$
   b) Either $A=0$ or $B=0$ (or both).
   c) If $\det A=-3$, then $B=0$.
   d) If B is invertible then $A=0$.
   e) There is a vector $V\neq 0$ such that $BAV=0$.

4. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V}=\mathbf{Y}_1$ and $A\mathbf{W}=\mathbf{Y}_2$, respectively, while $\mathbf{Z}\neq 0$ is a solution of the homogeneous equation $A\mathbf{Z}=0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.
   a) Find some solution of $A\mathbf{X}=3\mathbf{Y}_1$.
   b) Find some solution of $A\mathbf{X}=-5\mathbf{Y}_2$.
   c) Find some solution of $A\mathbf{X}=3\mathbf{Y}_1-5\mathbf{Y}_2$.
   d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X}=0$.
   e) Find *two* solutions of $A\mathbf{X}=\mathbf{Y}_1$.

f) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

g) If A is a square matrix, then $\det A =$?

h) If A is a square matrix, for any given vector $\mathbf{W}$ can one always find at least one solution of $A\mathbf{X} = \mathbf{W}$? Why?

5. a) If $r(\neq 0)$ is a rational number and x is irrational, show that both $r+x$ and rx are *irrational*.

   b) Prove that there is no rational number whose square is 12.

   c) Graph the points (x,y) in the plane $\mathbb{R}^2$ that satisfy $|y-x| > 2$.

6. a) Write the complex number $z = \dfrac{1}{a+ib}$ in the form $c+id$, where a, b, c are d are real numbers. Of course assume $a+ib \neq 0$.

   b) If $w \in \mathbb{C}$ satisfies $|w| = 1$, show that $1/w = \bar{w}$. [$\mathbb{C}$ is the set of complex numbers.]

7. Let $z, w, v \in \mathbb{C}$ be complex numbers.

   a) Show that $|z-w| \geq |z-v| - |v-w|$.

   b) Graph the points $z = x+iy$ in the complex plane that satisfy $1 < |z-i| < 2$.

   c) Let $z, w \in \mathbb{C}$ be complex numbers with $|z| < 1$ and $|w| = 1$. Show that

$$\left| \frac{w-z}{1-\bar{z}w} \right| = 1.$$

8. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees (+45 degrees means 45 degrees *counterclockwise*).

   b) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

   c) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

   d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

   e) Find the inverse of each of these maps.

9. Let the continuous function $f(\theta)$, $0 \leq \theta \leq 2\pi$ represent the temperature along the equator at a certain moment, say measured from the longitude at Greenwich.. Show there are antipodal points with the *same* temperature.

10. A certain function $f(x)$ has the property that $\int_0^x f(t)\,dt = e^x \cos x + C$. Find both f and the constant C.

11. If $b \geq 0$, show that for every real c the equation $x^5 + bx + c = 0$ has exactly one real root.

12. Let $p(x) := x^3 + cx + d$, where c, and d are real. Under what conditions on c and d does this has three distinct real roots? [HINT: Sketch a graph of this cubic. Observe that if there are three distinct real roots then there is a local maximum and the polynomial is positive there. What about a local min?].

13. Prove that the function $\sin x$ is not a polynomial. That is, there is no polynomial

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

with real coefficients so that $\sin x = p(x)$ for all real numbers x. In your proof you can use any of the standard properties of the function $\sin x$.

[Last revised: August 26, 2010]