Math508, Fall 2010 **Jerry L. Kazdan**

Problem Set 2

DUE: Thurs. Sept. 23, 2010. *Late papers will be accepted until 1:00 PM Friday.*

1. Let F be a field, such as the reals or the integers mod 7 and $x,\ y \in F$. Here $-x$ means the additive inverse of x.
   a) If $xy = 0$, then either $x = 0$ or $y = 0$ (or both are 0).
   b) Show that $(-1)x = -x$. [First I needed to prove that $0x = 0$.]
   c) Show that $(-x)(-y) = xy$.

2. Let F be an ordered field, such as the reals or rationals.
   a) If $x < y$, show that $x < \dfrac{x+y}{2} < y$.
   b) For each $x \in F$, if $x \neq 0$, then $x^2 > 0$.
   c) If $x^2 + y^2 = 0$, then $x = y = 0$.

3. Show that the field of complex numbers *cannot* be made into an *ordered* field. In other words, there is no possible definition $z \prec w$ of order that has the properties of an order relation. [HINT: show that in an ordered field, the equation $z^2 + 1 = 0$ has no solution.]

4. Let $f(x,y) := (x^2+y^2)^2 + y^4 + 2xy - x + 7y$. Note that $f(0,0) = 0$.
   Find some explicit number R so that if $x^2 + y^2 \geq R^2$, then $f(x,y) \geq 1$.
   MORAL: the global minimum of f, if one exists, is inside the disc $x^2 + y^2 \leq R^2$. [You are not asked to find the *best* R].

5. a) Find all non-empty bounded sets A such that $\sup A \leq \inf A$.
   b) Let A and B be sets of real numbers. If A is bounded above and B is bounded below, prove that $A \cap B$ is bounded.
   c) Given any two real numbers x, y with $x < y$, prove there is an irrational number between them.

6. Theorem 1 (page 7) in Hoffman shows that the completeness property implies the *least upper bound property*: Let A be a non-empty set of real numbers. If A, is bounded above, then there is a real number c that is the least upper bound of A (c might or might not be in A).

   Prove the converse: The least upper bound property implies the completeness. Moral: These two properties are equivalent.

7. The point of this problem is to give an example of an ordered field that does *not* have the Archimenian property.

   Consider the set $\mathcal{R}$ of rational functions $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and $q(x)$ not identically zero. The function $f(x)$ has a finite value everywhere except at a finite numnber of points (the zeroes of $q(x)$).

   a) It should be obvious that with the usual definitions of addition and multiplication, the set of rational functions is closed under addition and multiplication, that is, the sum and product of two rational functions ia also a rational function. Show that $\mathcal{R}$ forms a field.

   b) In $\mathcal{R}$, define the order $f > 0$ to mean that $f(x) > 0$ for all sufficiently large positive real x. Thus, if
   $$f(x) = \frac{a_0 + a_1 x + \cdots + a_k x^k}{b_0 + b_1 x + \cdots + b_n x^n}$$
   with $a_k \neq 0$ and $b_n \neq 0$, then $f > 0$ means $\dfrac{a_k}{b_n} > 0$. [This gives an algebraic definition of $f > 0$ that avoids defining "sufficiently large".] Then $f > g$ is defined to mean $f - g > 0$. Show that with this order relation, $\mathcal{R}$ is an ordered field.

   c) Show that this ordered field is *non-archemedean* by exhibiting two specific rational functions f and g with the property that there is *no* integer N such that $Nf > g$.

Bonus Problems (Due Sept 24)

B-1 A complex number is *algebraic* if it is a root of a polynomial $a_0 z^n + \cdots + a_n$ whose coeffients are all integers. Prove that the set of all algebraic numbers is countable. [HINT: For every positive integer N there are only finitely many equations with $n + |a_0| + \cdots + |a_n| = N$.]

Use this to prove there exist real numbers that are not algebraic.

[Last revised: September 17, 2011]