Math508, Fall 2010 **Jerry L. Kazdan**

Problem Set 3

DUE: Thurs. Sept. 30, 2010. *Late papers will be accepted until 1:00 PM Friday.*

1. Find all (complex) roots $z = x + iy$ of $z^2 = i$.

2. Let $x_n > 0$ be a sequence of real numbers with the property that they converge to a real number $c > 0$. Prove there is a real number $m > 0$ such that $x_n > m$ for all $n = 1, 2, \ldots$.

3. Let $x_k \in \mathbb{R}$, $x_k \neq 0$ be a sequence of real numbers. If $x_k \to c \neq 0$, show that $1/x_k \to 1/c$.

4. Calculate $\lim_{n\to\infty} \sqrt{n^2+n} - n$.

5. If $c > 0$, show that $\dfrac{c^n}{n!} \to 0$ as $n \to \infty$.

6. Let a_n be an increasing sequence of real numbers that is bounded above, so there is an M such that $a_n < M$ for all $n = 1, 2, 3, \ldots$. Show there is a real number A such that $a_n \to A$.

7. Let $p_k = (x_k, y_k) \in \mathbb{R}^2$, $k = 1, 2, \ldots$ be a sequence of points in the plane (with the usual Euclidean metric). Show that $\{p_k\}$ converges to $p = (x, y)$ if and only if $x_k \to x$ and $y_k \to y$.

8. [NEWTON] Let $A > 0$ and $x_1 > 0$. Define $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{A}{x_n}\right)$. The following steps show that $x_n \to \sqrt{A}$.
   a) Show that after the first term, the sequence $\{x_n\}$ is monotonically decreasing and that $x_2^2 \geq A$ (hence $x_n^2 \geq A$ for $n \geq 2$).
   b) Show the x_n converge to some real number L and, using the definition of x_n, that $L^2 = A$.

9. a) Give an example of a nested sequence of *open* intervals whose intersection is empty.
   b) Give an example of a sequence of closed intervals $J_{k+1} \supset J_k$, $k = 1, 2, \ldots$, whose union is the *open* interval $(-2, 2)$.

10. [From Homework Set 0] Let $z, w, v \in \mathbb{C}$ be complex numbers.
    a) Show that $|z - w| \geq |z - v| - |v - w|$.
    b) Graph the points $z = x + iy$ in the complex plane that satisfy $1 < |z - i| < 2$.

c) Let $z, w \in \mathbb{C}$ be complex numbers with $|z| < 1$ and $|w| = 1$. Show that

$$\left|\frac{w-z}{1-\bar{z}w}\right| = 1.$$

Bonus Problem (Due Oct. 1)

B-1 Given a real sequence $\{a_k\}$, let $C_n = \dfrac{a_1 + \cdots + a_n}{n}$ be the sequence of averages (*arithmetic mean*).

a) Give an example where the a_n's doesn't converge but the averages do converge.

b) If a_k converges to A, show that also C_n converges, and to A.

c) If the $a_k \geq 0$ and the averages converge, must the a_k's be bounded? (Proof or counterexample)

[Last revised: October 5, 2010]