**Math508, Fall 2010** **Jerry L. Kazdan**

# Problem Set 9

DUE: Thurs. Nov. 18, 2010. *Late papers will be accepted until 1:00 PM Friday*.

**Note:** We say a function is *smooth* if its derivatives of all orders exist and are continuous.

1. Let $f(x)$ be a smooth function for $x \ge 1$ with the property that $f'(x) \to 0$ as $x \to \infty$.
   a) Show that $f(n+1) - f(n) \to 0$ as $n \to \infty$.
   b) Compute $\lim_{n\to\infty} \left[\sqrt[5]{n+1} - \sqrt[5]{n}\right]$.

2. Find a continuous function $f$ and a constant $C$ so that $\displaystyle\int_0^{2x} f(t)\,dt = 2x\cos x + e^{4x} + C$.

3. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function.
   a) If $\int_0^1 f(x)\,dx = 0$, prove that $f(x)$ is positive somewhere and negative somewhere in this interval (unless it is identically zero).
   b) Use this to show that $\|f\|_1 := \displaystyle\int_0^1 |f(x)|\,dx$ is a norm on $C([0,1])$.
   c) Show that $C([0,1])$ with this norm is *not* complete.

4. Let $f(x) \in C([a,b])$. Show that
$$\exp\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right] \le \frac{1}{b-a}\int_a^b \exp[f(x)]\,dx$$
   [HINT: Use the inequality $e^u \ge 1+u$ where $u = f - \bar{f}$. Here $\bar{f} =$ average of $f = \frac{1}{b-a}\int_a^b f(x)\,dx$.]

5. [Hoffman, p. 143 #2] If $G(x)$ is Riemann integrable on $[a,b]$ and $F(x) = G(x)$ except at one point, show that $F$ is Riemann integrable and
$$\int_a^b F(x)\,dx = \int_a^b G(x)\,dx.$$
   This obviously extends to where $F(x) = G(x)$ except at a finite number of points.

6. a) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* continuous functions $g$, prove that $f(x) = 0$ for all $x \in [0, 1]$.
   b) If $f : [0, 1] \to \mathbb{R}$ is a continuous function with the property that $\int_0^1 f(x)g(x)\,dx = 0$ for *all* $C^1$ functions $g$ that satisfy $g(0) = g(1) = 0$, must it be true that $f(x) = 0$ for all $x \in [0, 1]$? Proof or counterexample.

7. a) If $V=(x,y,z)\in\mathbb{R}^3$ and $p\geq 1$, define $\|V\|_p := [|x|^p+|y|^p+|z|^p]^{1/p}$. Show that $\lim_{p\to\infty}\|V\|_p = \max\{|x|,|y|,|z|\}$.

   b) Let $f\in C([a,b])$ and for $p\geq 1$ recall the notation

$$\|f\|_\infty = \max_{x\in[a,b]}|f(x)| \quad \text{and} \quad \|f\|_p = \left[\int_a^b |f(x)|^p\,dx\right]^{1/p}$$

   Show that

$$\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$$

8. Let $f\in C([0,\infty])$ be a continuous function with the property that $\lim_{x\to\infty} f(x)=c$. Show that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)\,dx = c.$$

Bonus Problems (Due Nov 18)

B-1 Let $f\in C([0,1])$. Show that $\displaystyle\lim_{\lambda\to\infty}\int_0^1 f(x)\sin(\lambda x)\,dx = 0.$

B-2 [HÖLDER'S INEQUALITY] Let $p,q>1$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$.

   a) Show that $st\leq \dfrac{s^p}{p}+\dfrac{t^q}{q}$ for all $s,t>0$.
   [SUGGESTION: There are many ways to prove this. One is to show that for any $a>0$ and $s\geq 0$ the maximum of $h(s):=as-s^p/p$ occurs at $s=a^{1/(p-1)}$.]

   b) Use this to show that for any complex numbers $a_k$, $b_k$

$$\sum_{k=1}^n |a_k b_k| \leq \left[\sum_{k=1}^n |a_k|^p\right]^{1/p}\left[\sum_{k=1}^n |b_k|^q\right]^{1/q}.$$

   [SUGGESTION: First do the special case $\left[\sum_{k=1}^n |a_k|^p\right]^{1/p}=1$ and $\left[\sum_{k=1}^n |b_k|^q\right]^{1/q}=1$. Then reduce the general case to this special case.]
   If $p=q=1/2$ this is the Schwarz inequality.

   c) Similarly, show that for any continuous functions $f$, $g$

$$\int_a^b |f(x)g(x)|\,dx \leq \left[\int_a^b |f(x)|^p\,dx\right]^{1/p}\left[\int_a^b |g(x)|^q\,dx\right]^{1/q}.$$

d) Let $p,q>1$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$. and let $X:=(x_1,\ldots,x_n)\in\mathbb{R}^n$ and $f\in C([a,b])$. Use Hölder's inequality (above) to prove the triangle inequality for the norms

$$\|X\|_p := \Big[\sum_{k=1}^{n}|x_k|^p\Big]^{1/p} \quad \text{and} \quad \|f\|_p := \left[\int_a^b |f(x)|^p\,dx\right]^{1/p}.$$

B-3 (For those who have studied rings). Let $\mathcal{C}$ be the ring of continuous functions on the interval $0\le x\le 1$.

a) If $0\le c\le 1$, show that the subset $\{\,f\in\mathcal{C}\mid f(c)=0\}$ is a maximal ideal.

b) Show that *every* maximal ideal in $\mathcal{C}$ has this form.

[Last revised: November 7, 2014]