Advanced Analysis: Problem Set 1 (due Tues. Jan 18, 2005)

Math 509, Spring 2005 **Jerry L. Kazdan**

This first problem set should all be review of material you already know.

1. Let $f \in C^2([0,3])$ have the properties $f(0) = 3$, $f(1) = 2$, and $f(3) = 5$. Show there is at least one point $z \in [0,3]$ where $f''(z) \geq \text{const} > 0$ and give an estimate for this constant.

2. Let $f \in C^2([0,1])$ have the properties $f(0) = f(1) = 0$ and $f''(x) \geq 0$ for all $x \in [0,1])$. Show that $f(x) \leq 0$ for all $x \in [0,1]$.

3. Let $p > 1$ and define q by $1/p + 1/q = 1$. Prove the following inequality for all real x and y:
$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$
[REMARK: There are many (elementary) ways to show this; none of them are completely obvious.]

4. Let $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) \equiv 0$ for all $x \leq 0$. Show that $f \in \mathbb{C}^\infty(\mathbb{R})$.

5. Consider two smooth plane curves $\gamma_1, \gamma_2 \colon (0,1) \to \mathbb{R}^2$ that do not intersect. Suppose P_1 and P_2 are points on γ_1 and γ_2, respectively, such that the distance $|P_1P_2|$ is minimal. Prove that the straight line P_1P_2 is normal (that is, perpendicular) to *both* curves.

6. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 \leq p < \infty$, let $\|x\|_p := \left[\sum |x_j|^p\right]^{1/p}$. Also, define $\|x\|_\infty := \max_j |x_j|$. Show that
$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty.$$
[This justifies the notation $\|x\|_\infty$.]

7. Consider the linear space S of real sequences $x = (x_1, x_2, \ldots)$ with only a finite number of non-zero terms. Let $\|x\| := \max_j |x_j|$.
   a) Show that this is a norm on this space.
   b) Is this space complete with this norm? Justify your response.

8. Consider the space ℓ_1 of real sequences $x = (x_1, x_2, \ldots)$ with the property that $\sum |x_j|$ converge absolutely. Show that

$$\|x\| := \sum |x_j|.$$

defines a norm on this space and that with this norm the space is complete.

9. Newton's method for computing the square root of a positive real number a uses

$$x_{n+1} = \tfrac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Show that with *any* initial guess $x_0 > 0$ this sequence converges to $\sqrt{a}$. [There are several approaches].

10. Let c_n be a sequence of real numbers that converges to c.

    a) Show that their "average" (arithmetic mean), $\frac{1}{n}[c_1 + c_2 + \cdots + c_n]$, also converges to c. [SUGGESTION: First reduce to the simpler special case $c = 0$.]

    b) Give an example of a sequence that does *not* converge but whose arithmetic mean does converge.

11. Let $f(t)$ be a continuous function for $0 \le t < \infty$. If $\lim_{t\to\infty} f(t) = c$, show that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,dt = c.$$