# Math 509: Problem Set 4 (due Thurs. Feb. 10, 2005)

1. Let $A$ be an $n\times n$ symmetric matrix and $b\in\mathbb{R}^n$ a given vector. Show that if $x\in\mathbb{R}^n$ is a critical point of the function
$$f(x) := \tfrac{1}{2}\langle x, Ax\rangle - < b,x >,$$
then it is a solution of $Ax=b$.

2. POLYNOMIAL INTERPOLATION This problem is an extension of Taylor's Theorem with remainder to useful related situations.
   a) Let $f(x)$ be a smooth function for $a\le x\le b$. Say you approximate it by a polynomial of degree one $p(x)$ (a straight line) with the property that $p(a)=f(a)$ and $p(b)=f(b)$. Obtain the following estimate for the error $E(x)=f(x)-p(x)$ for $a\le x\le b$:
   $$f(x)-p(x) = \frac{f''(c)}{2!}(x-a)(x-b)$$
   for some $c$ in the interval containing $a$, $b$, and $x$. [SUGGESTION: Fix $x$ and think of it as a constant. For $z$ in the interval containing $a$, $b$, and $x$ let
   $$G(z) := f(z)-p(z)-A(z-a)(z-b),$$
   where the constant $A$ is picked so that $G(x)=0$. Then use Rolle's theorem to show that $A=\frac{f''(c)}{2!}$ for some $c$ ...]
   b) Generalize this to where $f$ is approximated by a polynomial $p(x)$ of degree $k$ that agrees with $f$ at the $k+1$ distinct points $x_0<x_1<\ldots<x_k$, so $p(x_j)=f(x_j)$. [As Step 0 of this you should show that given $k+1$ distinct points $x_0<x_1<\ldots<x_k$, there is a *unique* polynomial $p(x$ of degree at most $k$ such that $p(x_j)=f(x_j)$.]

3. [Rudin, p. 138 #2] Say $f\in C([a,b])$ and $f\ge 0$. If $\int_a^b f(x)\,dx=0$, show that $f(x)=0$ for all $a\le x\le b$.

4. Let $f(x)\in C([a,b])$ be a continuous function with the property that $\int_a^b f(x)h(x)dx=0$ for *any* function $h\in C([a,b])$. Show that $f(x)\equiv 0$.

   Does this conclusion still hold if $\int_a^b f(x)h(x)dx=0$ for *any* function $h\in C([a,b])$ that is zero at the end points, $h(a)=h(b)=0$? Proof or counterexample.

5. Let $a(x)$, $b(x)$, and $\varphi(x,t)$ be smooth functions of the real variables $x$ and $t$. If

$$F(x) := \int_{a(x)}^{b(x)} \varphi(x,t)\,dt,$$

find a formula for the derivative of $F$. To avoid unimportant issues you may assume that $a(x) < b(x)$.

6. Consider solving the partial differential equation

$$u_{x_1x_1} + 2u_{x_1x_2} + 4u_{x_2x_2} = 0$$

Find a linear change of variable of the form $y = Sx$, where $S = (s_{ij})$ is a $2\times 2$ invertible matrix so that in these new variables the equation has the simpler form

$$u_{y_1y_1} + u_{y_2y_2} = 0.$$

7. If $p > 1$ and $1/p + 1/q = 1$, recall that in Homework 1 you proved the inequality $uv \le u^p/p + v^q/q$ for all $u \ge 0$ and $v \ge 0$.

   a) If $f(x) \ge 0$ and $g(x) \ge 0$ are continuous functions with $\int_a^b f(x)^p\,dx = 1$ and $\int_a^b g(x)^q\,dx = 1$, show that $\int_a^b f(x)g(x)\,dx \le 1$.

   b) [Hölder] More generally, If $f(x)$ and $g(x)$ are continuous functions, show that

   $$\int_a^b |f(x)g(x)|\,dx \le \left[\int_a^b |f(x)|^p\,dx\right]^{1/p} \left[\int_a^b |g(x)|^q\,dx\right]^{1/q}.$$

   [SUGGESTION: in one line reduce this to the previous part.]

   c) Define $\|f\|_p$ as:

   $$\|f\|_p = \left[\int_a^b |f(x)|^p\,dx\right]^{1/p}.$$

   Use Hölder's inequality above to prove the *triangle inequality*

   $$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

8. [cf Rudin, p. 239 #12] If $0 < a < b$ and $Q$ is the square $0 \le s < 2\pi$, $0 \le t < 2\pi$ in $\mathbb{R}^2$, show that the following map from $Q$ to $\mathbb{R}^3$ describes the surface of a torus:

$$\begin{aligned} x(s,t) &= (b + a\cos s)\cos t \\ y(s,t) &= (b + a\cos s)\sin t \\ z(s,t) &= a\sin s \end{aligned}$$

Also, find and classify the critical points of $x(s,t)$.