Advanced Analysis: Problem Set 6 (due Thurs. Feb. 25, 2005)

(*Late papers will be accepted until 1 PM Friday*)

Math 509, Spring 2005 **Jerry L. Kazdan**

1. For which subsets of **R** does the series
$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$$
converge uniformly? (*Hint*: Sum the series!)

2. (Rudin, p. 166 #6) Prove that the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ converges uniformly in any bounded interval – but does not converge absolutely for *any* value of x.

3. (Rudin, p. 166 #7) For $n = 1, 2, \ldots$ and $x \in \mathbb{R}$, define $f_n(x) := \dfrac{x}{1+nx^2}$.
   a) Show that f_n converges uniformly to a function f.
   b) Show that $f'(x) = \lim_{n\to\infty} f'_n(x)$ is correct if $x \neq 0$ but false if $x = 0$.

4. Let $f(x)$ be a real-valued continuous function defined for all $x \geq 0$. If $\lim_{x\to\infty} f(x) = c$, show that the sequence
$$Q_n := \int_0^a f(x^n)\,dx, \qquad n = 1, 2, 3, \ldots$$
converges for any $a > 0$.

5. (Rudin, p. 169 #20) Let $f \in C([0,1])$ have the property that $\int_0^1 f(x)x^n\,dx = 0$ for all $n = 0, 1, \ldots$. Show that $f \equiv 0$. [SUGGESTION: Use the Weierstrass Approximation Theorem.]

6. Say one has a continuous function $f(x)$ defined for *all* real x with the property that there is a sequence of polynomials $p_k(x)$ that converge uniformly to f for all x. Thus, given any $\varepsilon > 0$, then for all sufficiently large k we have
$$\sup_{x\in\mathbb{R}} |f(x) - p_k(x)| < \varepsilon.$$
Show that $f(x)$ must itself be a polynomial.

7. a) Let $\{a_n\}$ be a sequence of real numbers with the property that

$$|a_{k+1} - a_k| \leq \tfrac{1}{2}|a_k - a_{k-1}|, \qquad k = 1, 2, \ldots\ .$$

Show that this sequence converges to some real number.

b) NOTATION: $\|\varphi\| = \sup_{0 \leq x \leq 1} |\varphi(x)|$ (so this is the uniform norm). Using this notation, let $\{u_n(x)\}$ be a sequence of continuous functions for $0 \leq x \leq 1$ with the property that

$$\|u_{k+1} - u_k\| \leq \tfrac{1}{2}\|u_k - u_{k-1}\|, \qquad k = 1, 2, \ldots.$$

Show that the $\{u_n\}$ converge uniformly to a continuous function.

8. Let $\varphi(x)$, $x \in \mathbb{R}$ be a smooth function with the following properties

i). $\varphi(x) > 0$ for $\|x\| < 1$, $\varphi(x) = 0$ for $\|x\| \geq 1$,

ii). $\int_{\mathbb{R}} \varphi(x)\, dx = 1$.

Let $\varphi_k(x) := k\varphi(kx)$. For a continuous function $f(x)$ with $f(x) = 0$ for x outside a compact set $\mathcal{K}$, define

$$f_k(x) := \int_{\mathbb{R}} f(t)\varphi_k(x-t)\, dt.$$

a) Show that $\varphi_k(x) = 0$ for $\|x\| \geq 1/k$, and $\int_{\mathbb{R}} \varphi_k(x)\, dx = 1$.

b) Show that the f_k are smooth functions.

c) Show that $\lim_{k\to\infty} f_k(x) = f(x)$, and that this convergence is uniform.

9. For each of the following give an example of a sequence of continuous functions. Justify your assertions. [A clear sketch may be adequate — as long as it is convincing].

a) $f_n(x)$ that converge to zero at every x, $0 \leq x \leq 1$, but *not* uniformly.

b) $g_n(x)$ that converge to zero at every x, $0 \leq x \leq 1$, but $\int_0^1 g_n(x)\, dx \geq 1$.

c) $h_n(x)$ converge to zero uniformly for $0 \leq x < \infty$, but $\int_0^\infty h_n(x)\, dx \geq 1$.