Advanced Analysis: Problem Set 7 (due Thurs. Mar. 17, 2005)

(*Late papers will be accepted until 1 PM Friday*)

Math 509, Spring 2005 **Jerry L. Kazdan**

1. Let $f \in C([-1,1])$ be an *even* function: $f(-x) = f(x)$. Show that in the uniform norm you can approximate f arbitrarily closely by an even polynomial.

2. Let $f \in C([0,1])$ show you can approximate f arbitrarily closely by polynomials of the special form $p(x) = a_0 + a_3x^3 + \cdots + a_{3k}x^{3k}$ containing only constants and the powers x^3, x^6, $\ldots$, x^{3k} for some k. [REMARK: This can be answered in one line.]

3. If $f \in C^1([0,1])$, use the Weierstrass approximation theorem to show that given any $\varepsilon > 0$ there is a polynomial $p(x)$ such that $\|f - p\|_{C^1} < \varepsilon$. Here we are using the C^1 norm
$$\|f\|_{C^1} := \max_{0\le x\le 1}|f(x)| + \max_{0\le x\le 1}|f'(x)|.$$

4. a) In a complete metric space M with distance $d(x,y)$, let $x_j \in M$ be a sequence that satisfies $\sum_j d(x_{j+1},x_j) < \infty$. Show that the x_j converge to an element of M.
   b) Give an example showing that if $d(x_{j+1},x_j) \to 0$, then the sequence x_j might not converge.
   c) Let $\|f\| = \max_{0\le x\le 2}|f(x)|$ for $f \in C([0,2])$. If $f_j \in C([0,2])$ $j = 1,2,\ldots$ satisfies
   $$\|f_{j+1} - f_j\| \le \frac{1}{j^2},$$
   show that the f_j converge uniformly in the interval $[0,2]$.

5. Let $f \in C([0,2])$. Show that in the uniform norm there is a continuous piecewise-linear function arbitrarily close to f. [A *piecewise linear* function $g(x)$ on the interval $[a,b]$ has the property that $[a,b]$ is partitioned into a finite number of sub-intervals and on each sub-interval $g(x)$ is a straight line segment.]

6. Let $\varphi(x)$, $x \in \mathbb{R}^n$ be a smooth function with the following properties
   i). $\varphi(x) > 0$ for $\|x\| < 1$, $\varphi(x) = 0$ for $\|x\| \ge 1$,
   ii). $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$.

Let $\varphi_k(x) := k^n\varphi(kx)$. For a continuous function $f(x)$ with $f(x) = 0$ for x outside a compact set $\mathcal{K}$, define

$$f_k(x) := \int_{\mathbb{R}^n} f(t)\varphi_k(x-t)\,dt.$$

a) Give an example of a function φ with these properties.

b) Show that $\varphi_k(x) = 0$ for $\|x\| \geq 1/k$, and $\int_{\mathbb{R}^n} \varphi_k(x)\,dx = 1$.

c) Show that the f_k are smooth functions.

d) Show that $\lim_{k\to\infty} f_k(x) = f(x)$, and that this convergence is uniform.

7. Given continuous function $h(x,y)$ and $f(x)$ for $0 \leq x \leq c$, $0 \leq y \leq c$ we seek a solution $u(x)$ of the *integral equation*

$$u(x) = f(x) + \int_0^c h(x,y)u(y)\,dy. \qquad (*)$$

as follows. Let $u_0(x) \equiv 0$ and define $u_k(x)$, $k = 1,2,\ldots$, recursively by the rule

$$u_{k+1}(x) = f(x) + \int_0^c h(x,y)u_k(y)\,dy.$$

a) Show that if $c > 0$ is sufficiently small, then the $u_k(x)$ converge uniformly for $0 \leq x \leq c$ to a continuous function $u(x)$ that satisfies the integral equation (*).

b) In the special case where $h(x,y) := \sum_{i=1}^N a_i(x)b_i(y)$ (where the functions a_i and b_i are, say, continuous), then equation (*) can be written as

$$u(x) = f(x) + \sum_{i=1}^N Q_i a_i(x), \quad \text{where} \quad Q_i := \int_0^c b_i(y)u(y)\,dy. \qquad (**)$$

With this observation, show that one can reduce (*) to a system of N linear algebraic equations:

$$Q_i = \gamma_i + \sum_{j=1}^N \alpha_{ij}Q_j,$$

where

$$\gamma_i := \int_0^c b_i(x)f(x)\,dx \quad \text{and} \quad \alpha_{ij} := b_i(x)a_j(x)\,dx.$$

Thus the γ_i and α_{ij} are regarded as known constants and the Q_i are the unknowns. [Suggestion: In (**) substitute the formula for u back into the formula for Q_i.]

c) In the special case where $h(x,y) \equiv 1$ and $f(x) \equiv 1$, solve equation (*) explicitly. From this, show that indeed for some value of c a solution may *not* exist.