Advanced Analysis: Problem Set 8 (due Thurs. March 24, 2005)

Math 509, Spring 2005 **Jerry L. Kazdan**

1. Let $f(x)$ and $K(x,y)$ be given continuous functions for $x,y \in [0,a]$. Consider the following linear *integral equation* for the continuous function $u(x)$:

$$u(x) = f(x) + \int_0^x K(x,y)u(y)\,dy \tag{1}$$

   a) If one works on some sufficiently small interval $0 < c < a$ using the function space $C([0,c])$ with the uniform norm show that this equation has a unique solution. [The choice of c will depend on $\max_{x,y\in[0,a]}|K(x,y)|$.]

   b) Show that, there is in fact a unique solution on the whole interval $[0,a]$. One method is to use the function space $C([0,a])$ with the modified norm:

$$\|u\|_\alpha := \max_{x\in[0,a]}|u(x)e^{-\alpha x}|,$$

   where the constant $\alpha > 0$ is chosen cleverly depending on $\max_{x,y\in[0,a]}|K(x,y)|$. Note that for any α this norm is equivalent to the uniform norm on $C([0,a])$.

2. Let $T : M \to M$ be a map of a complete metric space M to itself. If T is contracting, we proved that T has a unique fixed point.

   a) Show that one can generalize this to just assuming that for some $N \geq 1$ the composition $T^{\{N\}}$ is contracting. [A simple example is if T is a nilpotent matrix].

   b) Use this method to show that the integral equation in part a) of the previous problem has a unique solution on the whole interval $[0,a]$.

3. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map with the property that there is a constant $0 < c < 1$ such that

$$\|G(x) - G(y)\| \leq c\|x - y\| \quad \text{for all} \quad xy \in \mathbb{R}^n.$$

   a) Show that the map $F(x) := x + G(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is invertible.

   b) Show that the inverse $x = \Phi(y)$ of this map $y = F(x)$ satisfies

$$\|\Phi(y) - \Phi(\hat{y})\| \leq \frac{1}{1-c}\|y - \hat{y}\| \quad \text{for all } y, \hat{y} \in \mathcal{B}.$$

   In particular, this explicit estimate implies that the inverse, Φ is continuous.

4. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map with the property that its first partial derivatives are uniformly bounded for all points in $\mathbb{R}^n$.

   a) Show there is a constant k such that for all x and y in $\mathbb{R}^n$

   $$\|G(x) - G(y)\| \le k\|x - y\|.$$

   b) Let $F(x) := x + \gamma G(x)$. If the real constant $\gamma > 0$ is chosen sufficiently small, show that F is a diffeomorphism of $\mathbb{R}^n$.

5. Discuss the mapping $F : (x,y) \to (x^2 - y^2, 2xy)$ [cf Rudin, p. 241 #18].

6. [Rudin, p. 241 #19] Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

   can be solved for x, y, u in terms of z, for x, z, u in terms of y for y, z, u in terms of x, but *not* for x, y, z in terms of u.

7. [Rudin, p. 242 #23] Let $f(x,y_1,y_2) = x^2 y_1 + e^x + y_2$ and note that $f(0,1,-1) = 0$. Show that near $y_1 = 1$, $y_2 = -1$ there is a smooth function $x = g(y_1,y_2)$ with $g(1,-1) = 0$ so that $f(g(y_1,y_2), y_1, y_2) = 0$. Also, compute the gradient of g at $(1,-1)$.

8. Let $p(x) := (x-1)(x-2)\cdots(x-6) = x^6 - 21x^5 + \cdots$ and let $p(x,\varepsilon)$ be the polynomial obtained by replacing $-21x^5$ by $-(21+\varepsilon)x^5$. Let $x(\varepsilon)$ denote the perturbed value of root $x = 4$, so $x(0) = 4$. Compute the sensitivity of this root as one changes ε, that is, compute $dx(\varepsilon)/d\varepsilon\big|_{\varepsilon=0}$.

9. Say $f(x,y,z)$ and $g(x,y,z)$ are smooth functions and that one can solve the equations

$$f(x,y,z) = 0 \qquad g(x,y,z) = 0$$

   for y and z as functions of x. What is dz/dx?

10. a) Let $A(t) = \left[a_{ij}(t)\right]$ be a square matrix whose coefficients depend smoothly on a real parameter t. If $\lambda(0)$ is a simple eigenvalue (that is, its algebraic multiplicity is one), show that $\lambda(t)$ is a smooth functions of t for t sufficiently small.

b) If the above matrix $A(t)$ is self-adjoint with $A(0)v = \lambda(0)v$, derive the formula

$$\lambda'(0) = \frac{\langle v, A'(0)v\rangle}{\|v\|^2} \qquad \left(\text{here} \quad ' = \frac{d}{dt}\right).$$

11. Show that if A is a square matrix that is sufficiently close to the identity matrix, then it has a square root, that is, there is a matrix B with $B^2 = A$. Moreover this matrix B is unique if it is required to be near the identity matrix.

[Last revised: March 25, 2005]