Advanced Analysis: Problem Set 9 (due Tues. April 4, 2005)

Math 509, Spring 2005 **Jerry L. Kazdan**

1. For which value(s) of the constant c can the system of equations:

$$\begin{aligned} u(x,y,z) &= x + xyz \\ v(x,y,z) &= y + xy \\ w(x,y,z) &= z + cx + 3z^2 \end{aligned}$$

can be solved for x, y, z as smooth functions of u, v, w near $(1,1,0)$? Justify your assertion(s).

2. Let $y = f(x,u)$ and $z = g(x,u)$ be smooth functions with, say, $f(x_0,u_0) = y_0$ and $g(x_0,u_0) = z_0$.

   a) Under what condition(s) can one eliminate u from these equations to express z as $z = F(x,y)$ as a smooth function of x and y near $x = x_0$, $y = y_0$?

   b) Assuming this, then compute $\partial z/\partial x$ and $\partial z/\partial y$ in terms of the derivatives of f and g. To make this computation more specific, assume that

$$f_x(x_0,u_0) = 1, \quad f_u(x_0,u_0) = -2, \quad g_x(x_0,u_0) = -3, \quad \text{and} \quad g_u(x_0,u_0) = 4.$$

3. Let $f(x)$, $a \le x \le b$ be a smooth function.

   a) If $f(c) = 0$ for some $a \le c \le b$, show that

$$|f(x)| \le \int_a^b |f'(t)|\,dt \le \sqrt{b-a}\left[\int_a^b |f'(t)|^2\,dt\right]^{1/2}.$$

   and hence, using the uniform norm $\|f\|_{\text{unif}} := \max_{a\le x\le b}|f(x)|$,

$$\|f\|_{\text{unif}} \le \int_a^b |f'(t)|\,dt \le \sqrt{b-a}\left[\int_a^b |f'(t)|^2\,dt\right]^{1/2}.$$

   b) If $\int_a^b f(t)\,dt = 0$ (this replaces the assumption $f(c) = 0$), show that the above inequality still holds.

   c) Use the result of part b) to show that for *any* smooth f

$$\|f\|_{\text{unif}} \le \int_a^b \left[|f'(t)| + \frac{1}{b-a}|f(t)|\right]dt \le c\left[\int_a^b \left(|f'(t)|^2 + |f(t)|^2\right)\,dt\right]^{1/2},$$

   where c is a constant depending on $b-a$. [Suggestion: Apply the previous part to $g := f - \bar{f}$ where $\bar{f}$ is the average of f over the interval.]

4. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function.

   a) Show that $\lim_{\lambda\to\infty} \int_0^1 f(x)\sin(\lambda x)\,dx = 0$.

   b) Compute $\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x|\,dx$.

   c) (generalization) If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous with period P, show that

$$\lim_{\lambda\to\infty} \int_0^1 f(x)\varphi(\lambda x)\,dx = \overline{\varphi} \int_0^1 f(x)\,dx,$$

   where $\overline{\varphi} := \frac{1}{P}\int_0^P \varphi(t)\,dt$ is the average of φ over one period.

5. Give a direct proof of the Morse Lemma for a smooth function $z = f(x,y)$ of two variables. Assume

$$f(0,0) = 0, \qquad f'(0,0) = 0, \qquad f''(0,0) \text{ is positive definite}$$

   and show that near the origin there are new coordinates $u = u(x,y)$, $v = v(x,y)$ so that in these coordinates $z = u^2 + v^2$. [You may begin by assuming f already has the special form

$$f(x,y) = x^2 h_{11}(x,y) + 2xyh_{12}(x,y) + y^2 h_{22}(x,y),$$

   for smooth functions $h_{ij}(x,y)$ where $h_{11}(0,0) = 1$, $h_{12}(0,0) = 0$, and $h_{22}(0,0) = 1$].

[Last revised: March 30, 2005]