Math 509
May 1, 2007

Final Exam

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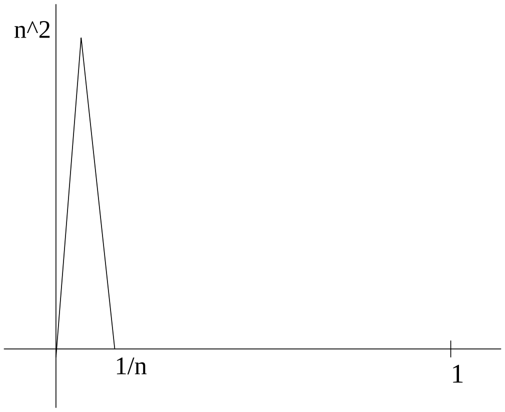
9:00 –11:00

DIRECTIONS This exam has three parts, Part A has 4 shorter problems (5 points each), Part B has 5 traditional problems (10 points each).
Closed book, no calculators – but you may use one $3'' \times 5''$ card with notes.

Part A: Shorter Problems (4 problems, 5 points each).

A–1. Give an example of a sequence of continuous functions $f_n(x)$, $0 \le x \le 1$, with $f_n(x) \to 0$ (pointwise) for all $x \in [0,1]$, but $\int_0^1 |f_n(x)|\,dx \to \infty$. A sketch is adequate.

Solution:



A–2. In $L_2(-1,1)$ with the standard inner product, show that any even function is orthogonal to any odd function (of course assume that the functions are integrable).

Solution: Let $h(x) = f(x)g(x)$, where $f(x)$ is even and $g(x)$ odd. Then $h(x)$ is odd. To show: $\int_{-1}^1 h(x)\,dx = 0$. This is clear geometrically. The computation is also easy:

$$I := \int_{-1}^1 h(x)\,dx = \int_{-1}^0 h(x)\,dx + \int_0^1 h(x)\,dx = -\int_{-1}^0 h(-x)\,dx + \int_0^1 h(x)\,dx.$$

But making the change of variable $t = -x$ we see that

$$-\int_{-1}^0 h(-x)\,dx = -\int_0^1 h(t)\,dt.$$

Thus $I = 0$.

A–3. Prove that the series $\displaystyle\sum_1^\infty \frac{(-1)^k \sin kx}{1+k^2}$ converges absolutely and uniformly for all real x.

Solution: Note that $\left|\dfrac{(-1)^k \sin kx}{1+k^2}\right| \le \dfrac{1}{1+k^2}$ for all x. Since $\dfrac{1}{1+k^2}$ converges, by the Weierstrass M-test the original series converges absolutely and uniformly for all x.

A–4. Let $u(x,y,t)$ be a solution of the heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ for $(x,\, y)$ in a bounded domain $D \in \mathbb{R}^2$ with the outer normal derivative $\nabla u \cdot N = 0$ on the boundary of D (here N is the unit outer normal vector field on the boundary).

If $Q(t) := \iint_D u(x,y,t)\,dx\,dy$, show that $\dfrac{dQ}{dt} = 0$ and hence that $Q(t) = Q(0)$.

Solution: By Green's theorem

$$\frac{dQ}{dt} = \iint_D u_t(x,y,t)\,dx\,dy = \iint_D \Delta u(x,y,t)\,dx\,dy = \int_{\partial D} \nabla u \cdot N\,ds = 0.$$

Part B: Traditional Problems (5 problems, 10 points each)

B–1. The following equations define a map $F : (x,y,z) \mapsto (u,v,w)$:

$$\begin{aligned} u(x,y,z) &= x + xyz^2 \\ v(x,y,z) &= xz^2 + y \\ w(x,y,z) &= 2x + cz + z^3 \end{aligned}$$

Clearly $F : (1,1,0) \mapsto (1,1,2)$. Write $p = (1,1,0)$ and $q = (1,1,2)$.

a) Compute the derivative $F'(p)$.

Solution:

$$F'(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & c \end{pmatrix}$$

b) For which value(s) of the constant c can the system of equations: can be solved for x, y, z as smooth functions of u, v, w near p? Justify your assertion(s).

Solution: By the inverse function theorem, the map is invertible (as a smooth map) if and only if $F'(p)$ is invertible. This is clearly true only for $c \neq 0$.

c) If c is one of these "good" values, let $G : (u,v,w) \mapsto (x,y,z)$ be the map inverse to F. Compute the derivative $G'(q)$ and use it to compute $\partial y(u,v,w)/\partial v$ at q.

Solution: By the inverse function theorem,

$$G'(q) = [F'(p)]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2/c & 0 & 1/c \end{pmatrix}.$$

Thus, $\partial y(u,v,w)/\partial v|_q = 1$.

B–2. In a Hilbert space $\mathcal{H}$, let $v_1, \ldots v_n$ be orthonormal vectors and $x \in \mathcal{H}$ a given vector.

a) Show there are scalars $a_1, \ldots, a_n$ and a $w \in \mathcal{H}$ with $w \perp \{v_1, \ldots, v_n\}$ so that

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + w.$$

Your work should exhibit a formula for the a_k in terms of x and $v_1, \ldots, v_n$.

Solution: Taking the inner product of the above equation with v_k we see that

$$\langle x,\, v_k \rangle = ipa_1 v_1 + a_2 v_2 + \cdots + a_n v_n + wv_k = a_k \langle v_k,\, v_k \rangle = a_k, \qquad k = 1, \ldots, n.$$

Now that we know the a_k, let $w = a - [a_1 v_1 + a_2 v_2 + \cdots + a_n v_n]$.

b) Show that $\|x\|^2 = |a_1|^2 + \cdots + |a_n|^2 + \|w\|^2$.

Solution: This is just the Pythagorean Theorem:

$$\|x\|^2 = \langle a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + w,\, a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + w \rangle$$

and observe that the orthogonality shows that the cross product term all are zero.

B–3. Let u and v be harmonic functions in a bounded (connected) region D with $u = f$ and $v = g$ on the boundary of D. If $f < g$, show that $u < v$.

Solution: Let $w = u - v$. Then w is harmonic and, by the maximum principle, has its maximum on the boundary. But on the boundary $w = f - g < 0$.

B–4. In homework you found that the Fourier series for $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi, \\ -1 & \text{for } -\pi \le x < 0. \end{cases}$ is

$$f(x) \sim \frac{2}{i\pi} \left[\left(\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \cdots \right) - \left(\frac{e^{-ix}}{1} + \frac{e^{-3ix}}{3} + \frac{e^{-5ix}}{5} + \cdots \right) \right].$$

Use this and the Parseval Theorem to compute

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Solution: Parseval's Theorem says that if $f(x) \sim \sum_{-\infty}^{\infty} c_k e^{ikx}$, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |c_k|^2.$$

Since $\int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2\pi$, this gives

$$1 = \frac{8}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \right]$$

so

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

B–5. Let $\varphi_k(x)$, $x \in \mathbb{R}$, be a sequence of smooth functions with the following properties

i). $\varphi_k(x) \geq 0$ for $|x| < 1/k$, $\qquad \varphi_k(x) = 0$ for $|x| \geq 1/k$,

ii). $\int_{\mathbb{R}} \varphi_k(x)\, dx = 1$.

For a continuous function $f(x)$ with $f(x) = 0$ for x outside a compact set $\mathcal{K}$, define

$$f_k(x) := \int_{\mathbb{R}} f(y)\varphi_k(x-y)\, dy.$$

Show that $\lim_{n\to\infty} f_k(x) = f(x)$, and that this convergence is uniform.

Solution:

$$f_k(x) - f(x) = \int_{\mathbb{R}} [f(x-y) - f(x)]\varphi_k(y)\, dy.$$

Since $f(x) = 0$ for x outside a compact set, it is uniformly continuous. Thus, for any $\epsilon > 0$ there is a $\delta > 0$ so that if $|y| < \delta$ then $|f(x-y) - f(x)| < \epsilon$. Consequently, if $1/k < \delta$, then

$$|f_k(x) - f(x)| \leq \int_{|y|\leq 1/k} |f(x-y) - f(x)|\varphi_k(y)\, dy \leq \epsilon \int_{|y|\leq 1/k} \varphi_k(y)\, dy \leq \epsilon.$$

This holds for all x so the convergence is uniform.