Math 509: Problem Set 4 (due Thurs. Feb 8, 2007)

1. For each of the following give an example of a sequence of continuous functions. Justify your assertions. [A clear sketch may be adequate — as long as it is convincing].
   a) $f_n(x)$ that converge to zero at every x, $0 \le x \le 1$, but *not* uniformly.
   b) $g_n(x)$ that converge to zero at every x, $0 \le x \le 1$, but $\int_0^1 g_n(x)\,dx \ge 1$.
   c) $h_n(x)$ converge to zero uniformly for $0 \le x < \infty$ with $h_n(x) = 0$ for $x \ge n$, but $\int_0^\infty h_n(x)\,dx \ge 1$.

2. Let $f \in C([0,2])$. Show that f can be approximated as as closely as one wishes in the uniform norm on $[0,2]$ by a step function.

3. Let $f \in C([-1,1])$ be an *even* function: $f(-x) = f(x)$. Show that in the uniform norm you can approximate f arbitrarily closely by an even polynomial. [REMARK: Show that every function $f(x)$ can be written as the sum of an even function, $f_{\text{even}}(x)$, and an odd function $f_{\text{odd}}(x)$: so $f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$. Here $f_{\text{even}}(x) = \frac{1}{2}[f(x) + f(-x)]$.]

4. Let $f \in C([0,1])$ show you can approximate f arbitrarily closely by polynomials of the special form $p(x) = a_0 + a_3x^3 + \cdots + a_{3k}x^{3k}$ containing only constants and the powers x^3, x^6, $\ldots$, x^{3k} for some k. [REMARK: Let $t = x^3$, that is, let $g(t) = f(t^{/}3)$. Then this problem can be answered in one line.]

5. If $f \in C^1([0,1])$, use the Weierstrass approximation theorem to show that given any $\varepsilon > 0$ there is a polynomial $p(x)$ such that $\|f - p\|_{C^1} < \varepsilon$. Here we are using the C^1 norm
$$\|f\|_{C^1} := \max_{0\le x\le 1} |f(x)| + \max_{0\le x\le 1} |f'(x)|.$$

6. Let $f(x)$ be a continuous function for $0 \le x \le 10$. Find all real numbers c for which $Q_c(f) := \lim_{n\to\infty} n^c \int_0^{10} f(x)e^{-nx}\,dx.$ exists. If the limit $Q_c(f)$ exists, compute it.

7. Let $f \in C([0,1])$. Show that $\lim_{n\to\infty} \int_0^1 f(x)\sin nx\,dx = 0$.

8. (Rudin, p. 169 #20) Let $f \in C([0,1])$ have the property that $\int_0^1 f(x)x^n\,dx = 0$ for all $n = 0, 1, \ldots$. Show that $f \equiv 0$. [SUGGESTION: Use the Weierstrass Approximation Theorem.]

9. Let $f \in C([0,2\pi]$ be periodic with period 2π.

   a) Show that given any $\varepsilon > 0$ there is a trigonometric polynomial:

   $$T(x) = a_0 + \sum_{n=1}^{N} [a_k \cos kx + b_k \sin nx]$$

   (where the a_k and b_k are constants), so that in the uniform norm on $[0,2\pi]$ we have $\|f - T\|_\infty < \varepsilon$. [SUGGESTION: Use the Stone-Weirestrass Theorem].

   b) Does the above conclusion still hold if we drop the assumption that f is periodic with period 2π? Why?

[Last revised:February 6, 2007]