Math 509: Problem Set 6 (due Tuess. March 13, 2007)

1. Let $f(x)$ and $K(x,y)$ be given continuous functions for $x,y \in [0,a]$. Consider the following linear *integral equation* for the continuous function $u(x)$:
$$u(x) = f(x) + \int_0^x K(x,y)u(y)\,dy \tag{1}$$
   a) If one works on some sufficiently small interval $0 < c \le a$ using the function space $C([0,c])$ with the uniform norm show that this equation has a unique solution. [The choice of c will depend on $\max_{x,y\in[0,a]}|K(x,y)|$.]
   b) Show that, there is in fact a unique solution on the *whole* interval $[0,a]$. One method is to use the function space $C([0,a])$ with the modified norm:
$$\|u\|_\alpha := \max_{x\in[0,a]} |u(x)e^{-\alpha x}|,$$
   where the constant $\alpha > 0$ is chosen cleverly depending on $\max_{x,y\in[0,a]}|K(x,y)|$.
   REMARK: For any α this norm is equivalent to the uniform norm on $C([0,a])$.

2. In the previous problem, use the observation that the contracting mapping principle is applicable if one knows that for some $N \ge 1$ the composition $T^{\{N\}}$ is contracting. Use this method to show that the integral equation in part a) of the previous problem has a unique solution on the whole interval $[0,a]$.

3. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map with the property that there is a constant $0 < c < 1$ such that
$$\|G(x) - G(y)\| \le c\|x - y\| \quad \text{for all} \quad xy \in \mathbb{R}^n.$$
   a) Show that the map $F(x) := x + G(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is invertible.
   b) Show that the inverse $x = \Phi(y)$ of this map $y = F(x)$ satisfies
$$\|\Phi(y) - \Phi(\hat{y})\| \le \frac{1}{1-c}\|y - \hat{y}\| \quad \text{for all } y, \hat{y} \in \mathcal{B}.$$
   In particular, this explicit estimate implies that the inverse, Φ is continuous.

4. Discuss the mapping $F : (x,y) \to (x^2 - y^2,\, 2xy)$ [cf Rudin, p. 241 #18].

5. [Rudin, p. 241 #19] Show that the system of equations

$$\begin{aligned} 3x+y-z+u^2 &= 0 \\ x-y+2z+u &= 0 \\ 2x+2y-3z+2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z, for x, z, u in terms of y for y, z, u in terms of x, but *not* for x, y, z in terms of u.

6. [Rudin, p. 242 #23] Let $f(x,y_1,y_2) = x^2y_1 + e^x + y_2$ and note that $f(0,1,-1) = 0$. Show that near $y_1 = 1$, $y_2 = -1$ there is a smooth function $x = g(y_1,y_2)$ with $g(1,-1) = 0$ so that $f(g(y_1,y_2),y_1,y_2) = 0$. Also, compute the gradient of g at $(1,-1)$.

7. Let $p(x) := (x-1)(x-2)\cdots(x-6) = x^6 - 21x^5 + \cdots$ and let $p(x,\varepsilon)$ be the polynomial obtained by replacing $-21x^5$ by $-(21+t)x^5$, with t small. Let $x(t)$ denote the perturbed value of root $x = 4$, so $x(0) = 4$.

   a) Show that $x(t)$ is a smooth function of t for all $|t|$ sufficiently small.

   b) Compute the sensitivity of this root as one changes t, that is, compute $dx(t)/dt)\big|_{t=0}$.

8. a) Let $A(t) = \left[a_{ij}(t)\right]$ be a square matrix whose coefficients depend smoothly on a real parameter t. If $\lambda(0)$ is a simple eigenvalue (that is, its algebraic multiplicity is one, so $\lambda(0)$ is a "simple" root of the characteristic polynomial), show that $\lambda(t)$ is a smooth functions of t for t sufficiently small.

   b) If the above matrix $A(t)$ is self-adjoint with $A(0)v = \lambda(0)v$, derive the formula

$$\lambda'(0) = \frac{\langle v, A'(0)v\rangle}{\|v\|^2} \qquad \left(\text{here} \quad {}' = \frac{d}{dt}\right).$$

Bonus Problem 6-a If A is a square matrix that is sufficiently close to the identity matrix, show that it has a square root, that is, there is a matrix B with $B^2 = A$. Moreover this matrix B is unique if it is required to be near the identity matrix.

Bonus Problem 6-b If $h(x,y) = x^2 - 2xy + 5y^2$, since then $h(x,y) = (x-y)^2 + 4y^2$, it is clear that under the change of coordinates $u = x - y$, $v = 2y$ we can write $h = u^2 + v^2$ as a sum of squares. Prove that one can do this near the origin for any smooth function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ with the properties that

$$f(0,0) = 0, \qquad f'(0,0) = 0, \qquad f''(0,0) \text{ is positive definite}.$$

[Here f' is the gradient and f'' the second derivative matrix.]

[Last revised:February 24, 2007]