

Math 509: Problem Set 7 (due Thurs. March 22, 2007)

1. For which value(s) of the constant c can the system of equations:

$$u(x, y, z) = x + xyz$$

$$v(x, y, z) = y + xy$$

$$w(x, y, z) = z + cx + 3z^2$$

can be solved for x, y, z as smooth functions of u, v, w near $(1, 1, 0)$? Justify your assertion(s).

2. Let $y = f(x, u)$ and $z = g(x, u, v)$ be smooth functions with, say, $f(x_0, u_0) = y_0$ and $g(x_0, u_0, v_0) = z_0$.

- a) Under what condition(s) can one eliminate x from these equations to express z as $z = F(y, u, v)$ as a smooth function of $y, u,$ and v near $x = u_0, y = y_0, v = v_0$?
- b) Assuming this, then compute $\partial z / \partial u$ and $\partial z / \partial y$ in terms of the derivatives of f and g . To make this computation more specific, assume that

$$f_x(x_0, u_0) = 1, \quad f_u(x_0, u_0) = -2, \quad g_x(x_0, u_0, v_0) = -3, \quad g_u(x_0, u_0, v_0) = 4 \text{ and } g_v(x_0, u_0, v_0) = -2.$$

3. Let $f(x), a \leq x \leq b$ be a smooth function.

- a) If $f(c) = 0$ for some $a \leq c \leq b$, show that

$$|f(x)| \leq \int_a^b |f'(t)| dt \leq \sqrt{b-a} \left[\int_a^b |f'(t)|^2 dt \right]^{1/2}.$$

and hence, using the uniform norm $\|f\|_{\text{unif}} := \max_{a \leq x \leq b} |f(x)|$,

$$\|f\|_{\text{unif}} \leq \int_a^b |f'(t)| dt \leq \sqrt{b-a} \left[\int_a^b |f'(t)|^2 dt \right]^{1/2}.$$

- b) If $\int_a^b f(t) dt = 0$ (this replaces the assumption $f(c) = 0$), show that the above inequality still holds.
- c) Use the result of part b) to show that for *any* smooth f

$$\|f\|_{\text{unif}} \leq \int_a^b \left[|f'(t)| + \frac{1}{b-a} |f(t)| \right] dt \leq c \left[\int_a^b (|f'(t)|^2 + |f(t)|^2) dt \right]^{1/2},$$

where c is a constant depending on $b - a$. [Suggestion: Apply the previous part to $g := f - \bar{f}$ where \bar{f} is the average of f over the interval.]

4. Say one has a real normed linear space whose norm comes from an inner product, so $\|x\|^2 = \langle x, x \rangle$. Show that the norm satisfies the *parallelogram identity*:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

and interpret this identity geometrically.

5. [Review from Math 241] Let \mathcal{P}_3 denote the space of real polynomials of degree at most three with the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx.$$

Find an orthonormal basis for this space.

6. [Review from Math 241] Consider the space of complex-valued continuous functions $C([- \pi, \pi])$ With the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(x)\overline{g(x)} dx,$$

and let T_N be the linear space spanned by e^{ikx} for $|k| \leq N$. Find the coefficients c_k so that

$$x^2 = \sum_{|k| \leq N} c_k e^{ikx} + h(x),$$

where $h(x)$ is orthogonal to T_N .

7. a) Let $f(x) \in C([0, 1])$. If $\int_0^1 f(x)x^n dx = 0$ for all $n = 0, 1, 2, \dots$, show that f must be identically zero.

b) Let $f(x) \in C([0, 1])$. If $\int_0^1 f(x)x^{2k} dx = 0$ for all $k = 0, 1, 2, \dots$, what can you conclude?

c) Let $f(x) \in C([-1, 1])$. If $\int_{-1}^1 f(x)x^n dx = 0$ for all $n = 0, 1, 2, \dots$, what can you conclude?

d) Let $f(x) \in C([-1, 1])$. If $\int_{-1}^1 f(x)x^{2k} dx = 0$ for all $k = 0, 1, 2, \dots$, what can you conclude?

8. a) Compute $\min_{a,b,c} \int_{-1}^1 |x^3 - a - bx - cx^2|^2 dx$.

b) Compute $\max \int_{-1}^1 x^3 h(x) dx$ where $h \in L^2(-1, 1)$ is subject to the restrictions

$$\int_{-1}^1 h(x) dx = \int_{-1}^1 xh(x) dx = \int_{-1}^1 x^2 h(x) dx = 0; \quad \int_{-1}^1 |h(x)|^2 dx = 1.$$

9. [Dual variational problems] Let V be a finite dimensional subspace of a Hilbert Space H and W its orthogonal complement. Recall that we can decompose any $x \in H$ uniquely as

$$x = x_V + x_W, \quad \text{where } x_V \in V, \text{ and } x_W \in W.$$

a) Show that $\max_{\{z \perp V, \|z\|=1\}} \langle x, z \rangle = \|x_W\|$.

b) Show that $\min_{v \in V} \|x - v\| = \|x_W\|$.

[Remark: *dual variational problems* are a pair of maximum and minimum problems whose extremal values are equal.]

Bonus Problem 7. Say one has a real normed linear space whose norm satisfies the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Define the bilinear form $B(x, y)$ by the rule (see Problem 4 above)

$$B(x, y) := \frac{1}{4} [\|x + y\|^2 - \|x - y\|^2].$$

- a) Show that $B(x, y)$ has all of the properties of an inner product including $B(x, x) = \|x\|^2$. [First observed by von Neumann]

MORAL: If a norm satisfies the parallelogram identity, then one can use it to define a compatible inner product; the parallelogram identity is both necessary and sufficient for the norm to arise from an inner product.

- b) Consider \mathbb{R}^2 with vectors $X = (x_1, x_2)$ having the p norm, $\|x\|_p := (|x_1|^p + |x_2|^p)^{1/p}$, $p \geq 1$. Use the result of the previous problem to show that if $p \neq 2$, this norm does *not* arise from an inner product.

[Last revised: March 12, 2007]