Comprehensive Examination **March 10, 1963**

①

Complex Variables

1. Find all the conformal mappings of the upper half-plane onto the interior of a semi-circle.

2. a) Use the calculus of residues to evaluate the integral $I = \int_0^\infty \frac{dx}{1+x^4}$.

   b) Use contour integration to sum the infinite series $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

3. a) State the *argument principle* for analytic functions.

   b) How is the argument principle used to find the number of complex zeroes of a polynomial?

4. a) Find the radius of convergence of the power series $f(z) = \sum_{n=0}^{\infty}(n+1)z^n$.

   b) Find the Taylor series expansion of the above function about the point $z=-1$. Where does it converge?

5. Find a bounded harmonic function in the unit circle $|z|<1$ that has boundary values 0 on the arc $|z|=1$, $0<\arg z<\pi$ and the boundary values -1 on the arc $|z|=1$, $\pi<\arg z<2\pi$.

6. Consider the function $w=u+iv=i\sqrt{\log z}$, $w(1)=i$. Show that u is continuous in $|z-1|\le 1$ and that $\iint_{|z-1|<1}(u_x^2+u_y^2)\,dxdy$ does *not* exist.

Complex Analysis

May 16, 1959

1. For which value(s) of the complex number α is the function $x^2-y^2+i\alpha xy$ an analytic function of $x+iy$?

2. The power series $\sum_{n=0}^{\infty} a_n z^n$ converges at $z=2i$. What can be said about its convergence at $z=2$? at $z=i$?, at $z=3$?

3. Match each function in the following list with the description on the right of its behavior at $z=0$

   (a). $z\sin\frac{1}{z}$,
   (b). $(\cot z)^3$,
   (c). $f(z)=\begin{cases}\frac{1-e^z}{z} & \text{for } z\neq 0,\\ 0 & \text{for } z=0,\end{cases}$
   (d), $\frac{\log(1+z)}{z^2}$
   (e). $e^{\frac{1}{z-1}}$

   (A). regular analytic,
   (B). removable singularity,
   (C). simple pole,
   (D). pole of higher order,
   (E). essential singularity

4. Suppose for $|z|\le 1$ that $f(z)$ is analytic and $|f(z)\le 1|$, and that $f(0)=f'(0)=0$. Prove that $|f(z)|\le|z^2|$ for $|z|\le 1$.

5. Evaluate the contour integral $\oint_C \frac{dz}{z^5-z^4}$ for the following contours described counterclockwise:

   a) C is the circle $|z|=\frac{1}{2}$.

   b) C is the circle $|z|=2$.

6. Map the first quadrant $\mathrm{Re}\{z\ge 0\}$, $\mathrm{Im}\{z\ge 0\}$ conformally onto the unit disk $|w|\le 1$ in such a way that 0 is mapped to $-i$, 1 is mapped to itself, and ∞ is mapped to -1.

Complex Analysis — September 21, 1959

1. Determine all the values, $a+bi$ of i^i.

2. Evanuate the contour integrals $\oint \frac{e^z-1}{z^2-z}\,dz$ and $\oint \frac{dz}{z^2-z}$ for the following contours described counter-clockwise:
   a) The circle of radius $\frac{1}{2}$ around the origin.
   b) The circle of radius 2 around the origin.

3. Classify the singularities of the following functions:
   a) $e^{1/z}$ at $z=0$.
   b) $e^{z+\sin z}$ at $z=\infty$.
   c) $\int_0^z e^{e^t}\cos t\,dt$ at $z=\pi$.
   d) $z^{3/2}$ at $z=0$.
   e) $\frac{\sin(\pi z)}{z}$ at $z=0$.

4. for what values of $|z|$ do the power series:
   (a). $\sum_{n=1}^{\infty}(\frac{z}{n})^n$, (b). $\sum_{n=0}^{\infty}(-1)^n z^n$, (c). $\sum_{n=0}^{\infty} n^{1/n} z^n$
   (A). converge, (B). converge uniformly, (C). not converge

5. What conditions must the complex numbers a, b, c, d satisfy in order that the transformation $w=\frac{az+b}{cz+d}$ map the upper half plane into itself?

6. Assume the analytic function $f(z)$ satisfies $|f(z)|\le M$ for $|z|\le 1$ and $f(0)\neq 0$. Let N be the number of zeroes of $f(z)$ in the disc $|z|<\frac{1}{3}$. Prove that:
$$N<\frac{1}{\log 3}\log\frac{M}{|f(0)|}.$$
   [SUGGESTION: Apply the maximum modulus principle to $g(z)=\frac{f(z)}{\prod_{j=1}^{N}\left(\frac{z-a_j}{1-\bar{a}_j z}\right)}$, where $f(a_j)=0$ with $|a_j|<\frac{1}{3}$.

Complex Analysis — May 19, 1960

1. If $\sum a_n z^n$ represents the function $\frac{\sin z}{z^2+2}$, whch of the following are true? (Briefly justify your assertion).
   (A). Converges for $z=1$.
   (B). Converges absolutely for $z=1$.
   (C). Converges absolutely for $z=2$.

2. Let $f(z)=\frac{\sqrt{z^2+1}}{z^3-1}\log\left(\frac{z}{z-2}\right)e^z$. Match its behavior at the given point with the list on the right.

(a). $z=i$	(A). regular analytic
(b). $z=1$	(B). simple pole
(c). $z=0$	(C). higher order pole
(d). $z=\infty$	(D). algebraic branch point
(e). $z=-i$	(E). logarithmic branch point
(f). $z=-1$	(F). essential singularity

3. Which of the following can represent the real part of an analytic function?
   (a). x^2+y^2, (b). $x+y$, (c). x^2-y^2

4. Evaluate $\oint_C \frac{dz}{z^3+1}$ on the circles (traversed counterclockwise)
   a) Centered at the origin, radius $1/2$.
   b) $\{z\in\mathbb{C}\mid |z+\frac{3}{2}|=1\}$.

5. Find a conformal map from the half-strip $\{(x,y)\in\mathbb{C}\mid 0<x<\infty,\ 0<y<1\}$ to the full strip $\{(u,v)\in\mathbb{C}\mid -\infty<u<\infty,\ 0<v<1\}$.

6. The function $f(z)$ is analytic for $|z|<1$. It also has the properties:
   (i), $|f(z)|\le 1$ on $\{|z|<1\}$, and (ii). $f(\pm\frac{1}{2})=0$.
   a) Give an example of such a function.
   b) Prove that $|f(0)|\le\frac{1}{3}$ (this is not sharp).

Complex Analysis Sept. 14, 1961

1. Evaluate the integrals: $(a).\ \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx$ $\quad (b).\ \lim_{a\to\infty} \int_{-a}^{a} \frac{dx}{x+i}$.

2. Which of the following are the real part of a single-valued analytic function in the region $\{1 < |z| < 2\}$?
   a) x^2+y^2.
   b) $\ln(x^2+y^2) - \ln[(x+\frac{1}{2})^2+y^2]$.
   c) $|z-2|^{1/2}\cos[\frac{1}{2}\arg(z-2)]$.

3. Let $f(z)$ be analytic in $\{|z| < 1\}$ and continuous in $\{|z| \le 1\}$. Assume that $|f(z)| \le 1$ on $|z| = 1$. Prove that $f(z) + 8z^2 - 2 = 0$ has two roots in the unit disk.

4. Let $f(z)$ be an entire function with the property that $|f(z| \le e^x$ throughout the complex plane. What can be said about $f(z)$?

5. Suppose $f(z)$ is continuous in $\{|z| \le 1\}$ and analytic in $\{|z| < 1\}$ with $|f(z)| \le M$ on the part of the boundary in the upper-half plane while $|f(z)| \le m$ on the part of the boundary in the lower-half plane. Show that the function $g(z) := f(z) + \overline{f(\bar{z})}$ is analytic and that $|f(z)| \le \frac{1}{2}(m+M)$ on the segment of the real axis in the unit disk.

6. Find the function $f(z)$ that maps the strip $\{\mathrm{Im}\,\{z\} < \pi/2\}$ conformally onto the unit disk with $f(0) = 0$ and $f(\frac{i\pi}{2}) = i$.

Examination II

Math. 609 November 10, 1970
Dr. Kazdan 2-3:30 P.M.

Directions: Do any 5 (out of 6). If you do more, the best 5 will be selected. One 3"x5" card with notes may be used. Closed book.

1. Evaluate $\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+1)(x^2+4)}\,dx$, $a > 0$.

2. Let $f = u + iv$ be analytic in $\{|z| \le 1\}$ with $f(z) = \sum_{n\ge 0} a_n z^n$. Show that, for $z = e^{i\theta}$,

   (a) $\int_0^{2\pi} \overline{f(z)}\, e^{-ik\theta}\, d\theta = 0$, $k = 1,2,\ldots$,

   and hence

   (b) $a_k = \frac{1}{\pi}\int_0^{2\pi} u(z)\, e^{-ik\theta}\, d\theta$, $k = 1,2,\ldots$.

   (c) (Borel) If, in addition, one knows that $f(0) = 1$ and $u(z) \ge 0$, prove that $|a_k| \le 2$, $k = 1,2,\ldots$.

3. Let f be analytic in $\{|z| \le 1\}$ except for a <u>simple</u> pole at $z = \frac{1}{2}$. If, in addition, one knows that $f(\frac{i}{2}) = 0$ and $|f(z)| \le 1$ for $|z| = 1$, prove that

$$|f(0)| \le 1 .$$

Moreover, give an example showing equality is possible.

4. Let f be a rational function with a pole of order N at $z = a$ and of order M at $z = b$ ($\neq a$), and no other poles (except possibly at ∞). Prove that f has the partial fraction decomposition

$$f(z) = \text{polynomial} + \sum_{k=1}^{N} \frac{\alpha_k}{(z-a)^k} + \sum_{k=1}^{M} \frac{\beta_k}{(z-b)^k},$$

where α_k and β_k are complex numbers.

5. Prove that the equation $z + e^{-z} = 3$ has precisely one root in the right half-plane $\{x > 0\}$, and, moreover, that the root is real.

6. Let f be analytic in $D = \{|z| < 1\}$. If

$$|f(\tfrac{1}{n})| < \frac{1}{2^n}, \; n = 1,2,\ldots,$$

prove that $f \equiv 0$ in D. (Remark: for fixed $N > 0$, $\lim\limits_{n\to\infty} \frac{n^N}{2^n} = 0$)

:h 609 - Complex Analysis
. Rosenberg

Final Examination
Monday, December 17, 1979

STRUCTIONS: Do all five problems. In problems 3 and 5, go on to part (b) en if you can't finish part (a). The point value of each problem is dicated; there are a total of 100 points.

(20pts.) Show that the exponential function is "essentially characterized by its functional equation". More precisely, suppose f is an entire analytic function, not identically zero, satisfying the identity

$$f(z+w) = f(z)f(w) \quad \text{for all } z,w \in \mathbb{C}.$$

Prove that there exists some constant $c \in \mathbb{C}$ such that $f(z) = e^{cz}$ for all z. (Hint: there are several methods. In any event, first determine $f(0)$. Then you can either show that f has a single-valued logarithm g satisfying

$$g(z+w) = g(z) + g(w)$$

or else that f satisfies the differential equation

$$f'(z) = f'(0)f(z) .)$$

(15pts.) Find an explicit harmonic function u in the unit disk $\{z: |z|<1\}$ that extends continuously to the boundary with boundary values $u(e^{i\theta}) = \sin^2\theta$.

. (a) (15pts.) Prove the following important strengthening of a theorem proved in class: Let Ω be a connected simply connected open subset of $\mathbb{C}$ and let f be meromorphic in Ω with residue 0 at every pole. Show that there exists a single-valued meromorphic function g in Ω with $g' = f$. (You may assume the residue theorem.)

(b) (10pts.) Show by examples that the hypotheses of simple connectivity and of zero residues are both necessary.

. (15pts.) Prove (by contour integration) the following formula needed for the theory of the Γ-function:

$$\text{if } 0 < \text{Re}\, z < 1, \; \int_0^\infty \frac{t^{z-1}}{1+t}\, dt = \frac{\pi}{\sin \pi z}.$$

(Explain how you are choosing branches for any multivalued functions.)

(over)

5. (a) (20pts.) Prove the following theorem: let f be bounded and holomorphic in the upper half-plane {z: Im z > 0}, and assume that for some $c \in \mathbb{R}$,

$$\lim_{y\to+\infty} f(c+iy) = d$$

exists. Show that for any $x \in \mathbb{R}$, $\lim_{y\to\infty} f(x+iy) = d$ (with the convergence uniform in x on compact intervals).

(Hint: apply Vitali's Theorem or Montel's Theorem to a collection of translates of f.)

(b) (5pts) What does the theorem in (a) say about "limits at the boundary" of bounded holomorphic functions in the unit disk?

Complex Analysis

September 19, 1960

1. Justify your answers to each of the following.
   a) Is $e^x(\cos y + i\sin y)$ an analytic function of $z = x+iy$?
   b) Is $e^x(\sin y + i\cos y)$ an analytic function of z?
   c) Is $e^x \sin y$ the real part of an analytic function of z?

2. Suppose $\sum_{n=0}^{\infty} a_n(z-1)^n$ is the power series expansion of $\frac{1}{\cos z}$ about the point $z = 1$. Does the series $\sum_{n=0}^{\infty} |a_n|$ converge or diverge? Justify your assertion.

3. Find two different Laurent series expansions of $\frac{1}{z(z-1)}$ in annuli centered at $z = 1$. What is the domain of convergence of each series?

4. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$ by the calculus of residues.

5. Map the half-disk $\{|z| < 1, \ \mathrm{Im}\, z > 0\}$ conformally onto the first quadrant: $\{\mathrm{Re}\, w > 0, \ \mathrm{Im}\, w > 0\}$. Is the mapping $w = w(z)$ unique?

6. Let $f(z)$ be an entire analytic function of z with the property that $\mathrm{Im}\, f(z) \geq 0$. Prove that $f(z)$ is a constant.