Mathematics 246
Examination

I. Next to each item in the first list place as many items from the second list as are appropriate in general.

1. $(z^2 + 1) / z$
2. Pe(z,1,2i) (The Weierstrass ℘ function with primitive periods 1, 2i).
3. $e^{Pe(z,1,2i)}$
4. cot z
5. sin z

A. meromorphic
B. rational
C. elliptic
D. doubly periodic
E. has an essential singularity at a finite point
F. has an essential singularity at ∞
G. has a simple pole at the origin
H. has a double pole at the origin
I. can be expressed as a convergent product of bounded functions with simple zeros.
J. can be expressed as a convergent sum of rational functions
K. bounded in $1 < |z| < 2$.

II. Find, where possible, (if impossible, indicate why) the single-valued analytic functions that map,

a) The annulus $1 \leq |z| \leq 2$ onto the annulus $2 \leq |z| \leq 4$ so that the circle $|z| = 2$ goes into itself.

b) The strip $1 \leq x \leq 2$ onto the strip $1 \leq x \leq 3$.

c) The annulus $5 \leq |z| \leq 10$ onto the annulus $1 \leq |z| \leq 5$.

(II. continued)

d) A domain D with the two finite boundary points a,b into a half plane.

III. Prove one of the following:

a) The Schwarz reflection principle for analytic continuation across a line.

b) Given that $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, Re $z > 0$ show that $\Gamma(z)$ can be continued analytically into the domain Re $z > -1$ and that $\Gamma(z) = \lim_{n\to\infty} \int_0^n (1-\frac{t}{n})^n t^{z-1} dt$ for n an integer, Re $z > 0$.

IV. Let R be a rectangle ABCD with sides a, b, and R' a rectangle A'B'C'D' with sides a, b' where $b \neq b'$. Prove that R cannot be mapped conformally into R' so that $A \to A'$, $B \to B'$, $C \to C'$ and $D \to D'$.

Can R be mapped onto R' so that $A \to A'$, $B \to B'$, $C \to C'$ and the mapping is conformal.

Established theorems may be used without proof.

MATHEMATICS 245A Section A

Test No. 1

Questions may be answered in any order.

1) Let the power series $\sum_0^\infty a_n z^n$, $\sum_0^\infty b_n z^n$, $\sum_0^\infty a_n b_n z^n$ have radii of convergence r, r' and R, respectively. (a) Show that $R \geq rr'$. (b) Give an example with $R > rr'$.

2) Prove that $(1 + z/n)^n \longrightarrow e^z$ normally for $n \to \infty$ (n an integer). (Note that e^z is defined by the power series).

3) Let f(z) be entire. Prove that

$$\frac{f(z + 2h) - 2f(z + h) + f(z)}{h^2} \longrightarrow f''(z)$$

normally as $0 \neq h \longrightarrow 0$. (Hint: use Cauchy's integral formula).

4) Given that $\dfrac{e^z}{(z-1)(z+2)} = \sum_{-\infty}^{+\infty} a_n z^n$ in a domain D, what are the possible values of a_{-2}?

5) Let f(z) be holomorphic for $0<|z|<\infty$. What can you say about f(z) if you are told that

(a) $\operatorname{Im} f(z) > 0$?

(b) $|z^3 f(z)| < 5$?

(c) $f(\frac{1}{n}) = 1 + (-1)^n$, n = 1, 2, 3, ...?

(d) $|f(z)| \leq e^{e^{|z|}}$ and $f(n) = 1 + (-1)^n$, n = 1, 2, ...?

(e) $|f(z)| \leq |z|^{-1000}$ and $f(\frac{1}{n}) = 0$, n = 1, 2, ...?

(f) $f''(z) + f(z) = 0$ for real rational values of $z \neq 0$?

(g) $|f(z)| \leq |f(2)|$ for $|z - 2| < \frac{1}{10}$?

(h) f(z) has no zeros and $|f(z)| \geq |f(2)|$ for $|z-2| < \frac{1}{10}$?

(i) $\int_{|z|=10} f(z)z^n\, dz = 0$ for n = -1, -2, ...?

(j) $|f(z) - f(2)| > 1$ for $|z - 2| > 1$?

Give reasons for your answers.

EXAMINATION I

Math. 410
Dr. Kazdan

Thurs., Feb. 12, 1970
10-11 A.M.

Directions: Closed book. Answer Part A and any 3 from Part B (if all 4 questions in Part B are done, the best 3 will be selected). Each Question is 25 points. Extra credit for more.

PART A

TRUE - FALSE. ANSWER ANY 12. Score = 2#correct - # wrong. If some question appears ambiguous, you should feel free to give a reason for your answer.

1. $\overline{\left(\frac{1}{1 - 2i}\right)} = \frac{1 - 2i}{5}$.

2. If $a \neq 0$, b, and c are complex numbers that satisfy $ab = ac$, then $b = c$.

3. Let A be the set in $\mathbb{C}$ where $|z - 1| \leq |z + 3|$. Let B be the set in $\mathbb{C}$ where $-x \leq 1$. Then A = B.

4. The function $f(z) = 2xy + i(x^2 - y^2)$ is analytic for all complex numbers $z = x + iy$.

5. The function $f(z) = e^{\pi}$ is analytic for every $z \in \mathbb{C}$.

6. There is an analytic function $f = u + iv$ with $u(x,y) = x^2 - 2y^2$.

7. If $\sum a_n (z-i)^n$ converges for $z = 2+i$, then it must converge for $z = 0$.

8. If $\sum a_n z^n$ converges for $|z| < 1$, then $\sum n a_n z^{n-1}$ also converges for $|z| < 1$.

9. If t is any real number, then $|e^{it}| = 1$.

10. $e^{i\pi/2} = i$.

11. $|e^{iz}| \leq 1$ if and only if $\operatorname{Im}\{z\} \geq 0$.

12. $|\sin z| \leq 1$ for all $z \in \mathbb{C}$.

13. If $\cos z = 0$, then z must be real.

14. If $e^z = e^w$, where $z, w \in \mathbb{C}$, then we must have $z = w$.

PART B

1. Find the set in the complex plane where the following series converge (please do not worry about convergence on the boundary of these sets).

   a) $\sum_0^\infty \frac{n(z+i)^{2n}}{2^n}$ b) $\sum_0^\infty \frac{(2z)^n}{n!}$ c) $\sum_0^\infty e^{nz}$

2. Let $v(x,y) = 3 - 2x + y - e^y \sin x$. Find an analytic function $f = u + iv$ whose imaginary part is v and with $f(0,0) = 1 + 3i$.

3. Let $f_n(z) = \frac{1}{z^n + 1}$. Prove that f_n converges uniformly

   to $f(z) = 0$ for $|z| \geq 2$. (Remark: $|a + b| \geq |a| - |b|$).

4. If $\sum a_n z^n$ converges in the disc $|z| < r$ for some $r > 0$, prove that

$$\sum \frac{a_n}{n!} z^n$$

converges for all $z \in \mathbb{C}$.

EXAMINATION II

Math. 410
Dr. Kazdan

Thurs., March 26, 1970
10-11 A.M.

Directions: Closed book. Answer Part A and any 3 from Part B (if all 4 questions in Part B are done, the best 3 will be selected). Each question is 25 points. Extra credit for more.

PART A

True - False. Answer any 8. Score = 3#correct - 2#wrong. If some question appears ambiguous, you should feel free to give a reason for your answer.

1. If f is analytic in the open set $A \subset \mathbb{C}$, then its complex derivative, f', is also analytic in A.

2. There is no polynomial $p(z)$ (except $p(z) \equiv$ constant) such that $|p(z)| \leq 10{,}000$ for all z with $\mathrm{Re}\{z\} > 5$.

3. $\int_{|z|=2} \frac{e^z}{z-3}\, dz = 0$.

4. $\int_{|z-1|=3} \sin(e^z)\, dz = 0$.

5. If f is an entire function such that $|f(z)| \leq 10^{10}$ for $|z| \geq 10{,}000{,}547$, then $f \equiv$ constant.

6. If f is analytic in $\{|z| \leq 1\}$ and $f(z_k) = \sin z_k$, where $z_k = \frac{-1}{k}$, $k = 1, 2, \ldots$; then $f(z) \equiv \sin z$.

For 7 - 9, let $\sum_0^\infty a_n z^n$ be the Taylor series expansion of the function $\frac{\cos z}{3 - z}$. Then:

7. the series converges when $z = 1$,

8. the series converges absolutely when $z = 1$,

9. the series diverges when $z = 4$.

PART B

1. Evaluate the contour integral

$$\frac{1}{2\pi i}\int_\gamma \frac{2e^z}{z(z-2)}\,dz,$$

where γ is the contour a). $|z| = 1$, counterclockwise,
b). $|z-1| = 5$, counterclockwise.

2. Let a function f have the properties

a). f is analytic in $\{|z| \le 1\}$,
b). $|f(z)| \le 1$ for $|z| = 1$,
c). $f(1/3) = 0$.

Prove that $|f(\frac{1}{2})| \le 1/5$.

3. Let $h(t)$ be continuous for $0 \le t \le 1$ and define F by

$$F(z) = \int_0^1 h(t)\cos(tz)\,dt,$$

where z is a complex number. Prove that

a). F is an entire analytic function,

b). $|F(x+iy)| \le M\cosh y$, where M is a bound on h, i.e., $|h(t)| \le M$ for $0 \le t \le 1$ (recall $\cosh s = \frac{1}{2}(e^s+e^{-s})$).

4. Let f be analytic in the disc $\{|z| \le r\}$ and $f(z) \ne 0$ for $|z| = r$. If f has a zero of order n at the point z_0, $|z_0| < r$ (so $f(z) = (z-z_0)^n h(z)$, where $h(z)$ is analytic in $\{|z| \le r\}$ and $h(z_0) \ne 0$) but $f(z) \ne 0$ for all $z \ne z_0$, prove that

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{f'(z)}{f(z)}\,dz = n.$$

EXAMINATION III

Math. 410 — Thurs. April 23, 1970
Dr. Kazdan — 10-11 A.M.

Directions: Closed book. One 3 × 5 card with notes may be used.

PART A

Short answer. Answer any 10 (out of 13). 5 points each. Just write the answer in your blue book. Justification is not required.

1. The function $f(z) = \frac{1}{(z-1)(z-3)}$ has ______ different Laurent expansions about $z = 1$ (each expansion converging in a different region).

2. Exhibit a polynomial $p(z)$ such that

$$f(z) = \frac{p(z)}{1-\cos z}$$

has a simple pole at $z = 0$.

3. The residue of $f(z)$ of #1 at $z = 3$ is __________.

4. Let f be as in #1. Then the change in the argument of f as z traverses the circle $|z| = 2$ once counterclockwise is ___π___.

For 5 - 9, match the function on the left with the description of its behavior from the list on the right.

5. $\csc^2 z$ at $z = 0$	A. simple pole
6. $\tan z$ at $z = \pi/2$	B. pole of higher order
7. $2z \cot z$ at $z = 0$	C. regular analytic
8. $e^{\sin z}$ at $z = i$	D. removable singularity
9. $\cos \frac{1}{z}$ at $z = 0$	E. essential singularity

For 10 - 13, let f be analytic for $\{0 < |z| < \infty\}$. Using the list A - D above on the right, describe the behavior of f at $z = 0$ if you are told that

10. $|z^3 f(z)| \le 5$ for all $\{|z| \le 7\}$

11. $\int_{|z|=13} f(z) z^n \, dz = 0$, $n = -1, -2, \ldots$.

12. $f(i) = 1$ and $f\left(\frac{1}{n}\right) = 0$, $n = 1, 2, \ldots$

13. $|f(z)| \le |\sin z|$ if $\{|z| \le 1\}$

PART B

Answer any 2 (out of 3). 25 points each.

1. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^4} \, dx ,$$

2. The function

$$\frac{1}{z(z-2)^2}$$

has several Laurent series in annuli centered about $z = 2$. Describe all these annuli and find the coefficient of $(z-2)^{-2}$ for each of these Laurent series.

3. Let $\varphi(z)$ be analytic in $D = \{|z| \le 1\}$ and assume that $|\varphi(z)| \le 2$ on $\{|z| = 1\}$. How many roots does the function

$$h(z) = \varphi(z) + 5z^3 - 2$$

have in the disc D? Justify your assertion.