50. Find a conformal map of the half disk $\{|z|<1,\ y>0\}$ onto the unit disk $\{|w|<1\}$.

51. Prove that the function $f: z \mapsto e^z - z$ has an infinite number of zeroes in the complex plane.

52. Let p be a polynomial of degree at least two, all of whose roots lie in $\{|z|<R\}$. Prove

$$\oint_{|z|=R} \frac{1}{p(z)}\,dz = 2.$$

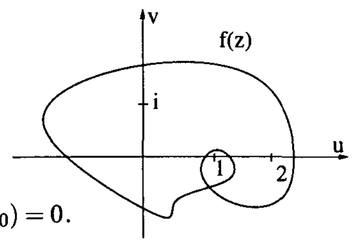
53. Give an example of a rational function $h(z)$ with $\dfrac{1}{2\pi i}\displaystyle\oint_{|z|=1} h(z)\sin z\,dz = 0$.

54. Let f be single-valued and analytic in $\{a<|z|<\frac{1}{a}\}$ and let $f(z)=\sum_{-\infty}^{\infty} a_n z^n$ be its Laurent expansion.

a) If $f(\frac{1}{\bar z}) = \overline{f(z)}$ prove that f is real for $|z|=1$ and that $a_{-n} = \bar a_n$.

b) Conversely, if f is real for $|z|=1$ and $a_{-n}=\bar a_n$, is it necessarily true that $f(\frac{1}{\bar z}) = \overline{f(z)}$? Proof or counterexample.

55. Assume $f = u+iv$ is analytic in $D=\{|z|\le 1\}$ so $f: D \to \mathbb{C}$. In the figure, the curve shows the path of $f(z)$ as z goes once around ∂D (the boundary of D). True or False — and why.



a) $f(\partial D) = \partial[f(D)]$.

b) There is precisely one point $z_0 \in D$ such that $f(z_0)=0$.

c) f is 1-to-1 on D.

d) There are three distinct points z_1, z_2 in D with $f(z_1)=f(z_2)$.

e) There are three distinct points z_1, z_2, z_3 in D with $f(z_1)=f(z_2)=f(z_3)$.

f) As z traverses ∂D counterclockwise, then $w=f(z)$ also traverses $f(\partial D)$ counterclockwise.

56. a) Consider the set of bounded analytic functions in $\{|z|<1\}$ with the norm

$$\|f\| = \sup_{\{|z|<1\}} |f(z)|$$

Do these form a Banach space? Why?

b) What about the functions analytic in the *closed* unit disk with the same norm?

57. Consider the *Dirichlet series* $\sum_{n\ge 1} \frac{a_n}{n^z}$, where the a_n and z are complex numbers. Let A be the points in the complex plane where this series converges absolutely.

a) Prove that *one* of the following holds:

(a) $A=\mathbb{C}$, (b) $A=$ empty set, (c) $A=\{z \mid \operatorname{Re} z > c\}$ for some real c.

b) Give examples showing that all three cases can occur.

58. If $f\in L^1(\mathbb{R})$ and $f(t)=0$ for all $t<0$, define

$$F(z) = \int_{-\infty}^{\infty} f(t)e^{itz}\,dt.$$

Prove that F is defined and holomorphic in the upper half plane $\operatorname{Im} z>0$.

59. Let $u(x,y)$ be positive and harmonic for all $(x,y)\in\mathbb{R}^2$. Prove that u must be a constant.

60. Define the differential operators $\partial/\partial z$ and $\partial/\partial\bar z$ by the rules:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad\text{and}\quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

a) Show that the Cauchy-Riemann equations for an analytic function $f=u+iv$ can be summarized by $\partial f/\partial\bar z = 0$.

b) Compute $\partial^2/\partial z\partial\bar z$ and use this to show that if $u(x,y)$ is a real harmonic function,then $\partial u/\partial z$ is an analytic function. [This construction is the simplest way to go from a real harmonic function to an analytic function.]

61. Let $f(x,y): \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map. Using the notation of the previous problem, say there is a real function $\mu(x,y)$, $0\le\mu\le 1$ such that

$$\left|\frac{\partial f}{\partial \bar z}\right| \le \mu \left|\frac{\partial f}{\partial z}\right|$$

for all (x,y). Show that the Jacobian matrix of f has positive determinant.

62. Let f be holomorphic in $\{|z| \le 1\}$ with $\max_{|z|=1}|f(z)| \le M$. Assume $f(0) = f'(0) = f''(0) = 0$. Show that
$$|f(z)| \le M|z|^3 \qquad \text{for all } |z| \le 1.$$

63. Let $p(z)$ be a polynomial of degree n and let $M(r) := \max_{|z|=r}|p(z|$. Show that if $0 < r < R$ then
$$\frac{M(r)}{r^n} \ge \frac{M(R)}{R^n}.$$

64. Let f and g be analytic in *DISK*. If neither f nor g are zero in the disk and $|f(z)| = |g(z)|$ on $|z| = 1$, what can you conclude about the relationship of f and g?

65. Let $Q \subset \mathbb{C}$ be open with $A := \{(x,y) \mid -1 \le x \le 1\} \subset Q$. If f is analytic on $Q - A$ and continuous on Q, prove that f is analytic on all of Q.

66. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence 1. If all the Taylor coefficients satisfy $a_n \ge 0$, prove that f is not analytic at $z = 1$. [SUGGESTION: Consider the power series about $z = 1/2$.]

HARNACK Assume the functions $f_n = u_n + iv_n$ are analytic in a domain $\omega \subset \mathbb{C}$. If the u_n converge uniformly in compact subsets of Ω and the numbers $v_n(c)$ converge for some point $c \in \Omega$, show that the functions f_n converge uniformly on compact subsets of Ω.

67. In the complex plane, let L be a straight line and p, q distinct points that lie on the same side of L. Prove that one can construct using ruler and compass all the circles throughj p and q having L as a tangent line. [HINT: Use conformal mapping].

68. Let A_r be the annulus $A := \{1 < |z| < r\}$. If there is a univalent conformal map from A_r to A_2, prove that $r = 2$.

69. Let $\Omega \subset \mathbb{C}$ be any connected open set. Prove there is a function f that is analytic in Ω and has the boundary of Ω as a natural boundary.

70. Let f be a non-constant meromorphic function with two periods, ω_1 and ω_2 such that ω_1/ω_2 is not real. Let P be the parallelogram with vertices 0, ω_1, ω_2, and $\omega_1 + \omega_2$, where 0 and the open line segments $(0, \omega_1)$, and $(0, \omega_2)$ but no other boundary points belong to P. Show that
$$2 \le \#\text{ zeroes of } f \text{ in } P = \#\text{ poles of } f \text{ in } P.$$

71. Let $\Omega \subset \mathbb{C}$ be a domain that is *not* simply connected. If $c \notin \bar{\Omega}$, prove that $f(z) = 1/(z-a)$ cannot be approximated arbitrarily (in the uniform norm) by a polynomial.

72. Let $A_2(\Omega)$ be the set of functions that are analytic in a region $\Omega \subset \mathbb{C}$ and are in $L^2(\Omega)$ with the norm
$$\|f\|^2 := \iint_\Omega |f(z)|^2\,dx\,dy.$$

a) If $B = \{|z - z_0| \le r\}$ is a closed ball in Ω, prove the *solid mean value property*
$$f(z_0) = \frac{1}{\pi r^2}\iint_B f(z)\,dx\,dy$$
and
$$|f(z_0)|^2 \le \frac{1}{\pi r^2}\|f\|^2,$$
where r is the distance from z_0 to the boundary of Ω.

b) Prove that A_2 is a (complete) Hilbert space with the obvious inner product.

c) If $f(z) = \sum_{n \ge 0} a_n (z - z_0)^n$ in B, show that
$$\iint_B |f(z)|^2\,dx\,dy = \pi \sum_{n \ge 0} |a_n|^2 \frac{r^{2n+2}}{n+1}.$$
Note this again proves the inequality in part a).

d) If $\Omega = \{|z| < 1\}$, show that $\varphi_k(z) := z^k$, $k = 0, 1, \ldots$ are orthogonal in $A_2(\Omega)$.

e) If $z_0 \in \Omega$ and Ω is bounded, prove that the functional $\ell(f) := f(z_0)$ is a bounded linear functional defined on $A_2(\Omega)$.

f) If $\Omega = \{|z| < 1\}$, use part d) to find a function $h \in A_2(\Omega)$ so that $\ell(f) = \langle f, h\rangle$. Here ℓ is from part d) and we use the inner product of $A_2(\Omega)$.

g) If $z_0 \in \Omega$ and Ω is bounded, prove that the functional $\ell(f) := f^{\{k\}}(z_0)$ is a bounded linear functional defined on $A_2(\Omega)$. (Here $f^{\{k\}}$ is the k^{th} derivative.)

h) Let $a_1, a_2, \ldots, a_n$ be distinct points in Ω and let
$$\mathcal{A}(\Omega) := \{f \in A_2(\Omega) \mid f(a_j) = 0,\ j = 1, 2, \ldots, n\}.$$
Prove that $\mathcal{A}(\Omega)$ is a (closed) subspace of $A(\Omega)$.

73. Assume the disk $A = \{|z - a| \le r\}$ is inside the unit disk $D = \{|z| < 1\}$. Find a univalent conformal map of the annular region between A and D onto some annulus of the form $\{1 < |w| < R\}$.

74. Assume f is analytic in $\{|z| \leq R\}$. Show that for $|z| < R$:

$$\overline{f(0)} = \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{\overline{f(\zeta)}}{\zeta - z} d\zeta.$$

$$f(z) + \overline{f(0)} = \frac{1}{i\pi} \oint_{|\zeta|=R} \frac{u(\zeta)}{\zeta - z} d\zeta.$$

$$f(z) - f(0) = \frac{1}{i\pi} \oint_{|\zeta|=R} \left[\frac{1}{\zeta - z} - \frac{1}{\zeta}\right] u(\zeta)\, d\zeta = \frac{1}{\pi} \int_0^{2\pi} \frac{z}{(Re^{i\theta} - z)} u(Re^{i\theta})\, d\theta.$$

$$f^{(k)}(z) = \frac{k!}{i\pi} \oint_{|\zeta|=R} \frac{u(\zeta)}{(\zeta - z)^{k+1}} d\zeta.$$

75. Let $\Omega \in \mathbb{C}$ be the intersection of the disks $|z-1| < 2$ and $|z+1| < 2$.

a) Find an injective conformal map from Ω to the unit disk.

b) Is there a conformal map that maps points on the imaginary axis in Ω to the interval between $\pm i$ on the imaginary axis? If so, is this map uniquely determined?

76. Let $\Omega \in \mathbb{C}$ be a (connected) simply connected open set that is symmetric under the map $z \longrightarrow \bar{z}$ (symmetric across the real axis).

a) Can you always find a conformal map from Ω to the open unit disk that maps the real points in Ω to points on the real axis?

b) Can you find a map f with the property that $f(\bar{z}) = \overline{f(z)}$?

Mathematics 630 FINAL EXAMINATION May 3, 1967

SUBMIT ANSWERS TO ANY EIGHT PROBLEMS.

1. If f is analytic on D and z_o is in D prove that $|f^{(n)}(z_o)| \geq n!n^n$ cannot hold for all n.

2. Show that if f is analytic on D and $|f(z)|$ is constant on the boundary of a bounded connected subset E of D then f is constant or f has a zero in E.

3. Prove that $w = \int_o^z \frac{dt}{\sqrt{1-t^4}}$ maps the unit disk onto a square.

4. Let f be a single-valued analytic function on a simply connected region D. Let C be any simple closed path in D having the origin inside. Prove that $\frac{1}{2\pi i}\int_C f'(z) \operatorname{Log} z \, dz = f(z_o) - f(0)$ where z_o is the initial and terminal point of integration.

5. (f_n) is a sequence of functions analytic on D. Prove that the uniform convergence of $\Sigma_{n=1}^{\infty} |f_n(z)|$ on every compact subset of D implies the same of $\Sigma_{n=1}^{\infty} |f_n^{(k)}(z)|$, $k = 1,2,\ldots$.

6. Show that $\pi \tan \pi z = \Sigma_{n=o}^{\infty} \frac{2z}{(n+\frac{1}{2})^2 - z^2}$.

7. Find the first four non-zero terms of the Laurent expansion of $\frac{e^z}{z^4(1-z^2)}$ on $0 < |z| < 1$.

8. Verify that a) $\int_o^{\infty} \frac{\cos^2 x \, dx}{(x^2+1)^2} = \frac{\pi}{8}(1+3e^{-2})$

   b) $\int_o^{\infty} \frac{x^{\alpha-1} dx}{x+1} = \frac{\pi}{\sin \alpha\pi}$ for $0 < \alpha < 1$.

9. Show that the functions z^n, n a non-negative integer form a normal family in $|z| < 1$, also in $|z| > 1$ but not in any region that contains a point of the unit circle.

10. Give a non-trivial example to illustrate the monodromy theorem. Give a non-trivial example to show that the conclusion of the theorem may be false if the assumption that the continuation paths are homotopic is dropped. Discuss the converse of the theorem.

Math. 213 b Homework Due March 28, 1966

1. Prove that the maximum modulus $M(r)$ of an entire function which is not a polynomial satisfies

$$\lim_{r \to \infty} \frac{\log M(r)}{\log r} = \infty.$$

2. What is the order of the functions

(i) ze^{3z} (ii) $e^z \cos z$

(iii) $\cos \sqrt{z}$ (iv) $\int_0^1 e^{zt^2}\, dt$

3. Prove that $\prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)}\right) = 2.$

4. Use Jensen's formula to find the value of

$$\frac{1}{2\pi} \int_0^{2\pi} \log |\sin z|\, d\theta$$

over the circle $|z| = r$.