Math 629: Topics in Algebraic Geometry – Algebraic Stacks – Spring 1999 A. Kresch
Lecture 1, January 13, 1999

- Idea of proof of irreducibility of M_g in arbitrary characteristic.

 M_g, $g \geq 2$, is the coarse moduli space of genus g curves over an alg. closed field k.

 $k = \mathbb{C}$. Teichmüller construction. M_g is a connected, normal variety.
 $\Rightarrow M_g(k)$ is irreducible for any field k of char 0.

 Fulton: Hurwitz scheme $H^{g+1,4g}$ covers $M_g(k)$ (char $k \neq 2$)
 could prove $H^{g+1,4g}$ connected when char $k > g+1$
 $\Rightarrow M_g(k)$ is irred for k of char $> g+1$.

 What about general k?

 Deligne and Mumford (1969): two proofs of irreducibility in arbitrary characteristic
 Proof 1: classical methods, inductive argument using stable reduction
 Proof 2: based on stacks
 $\overline{\mathcal{M}}_g \xrightarrow{f} \operatorname{Spec}\mathbb{Z}$
 Deformation theory $\Rightarrow f$ smooth;
 stable reduction theorem $\Rightarrow f$ proper;
 $f_*\mathcal{O}_{\overline{\mathcal{M}}_g}$ coherent, free, rank 1, so $\overline{\mathcal{M}}_g \times \operatorname{Spec} k$ connected for any k

- Schemes – quick review

- Schemes as spaces

 Definition: A **Zariski space** is a sheaf of sets for the Zariski topology on schemes

 concretely: a presheaf (a functorial assignment, to every scheme T, a set $Z(T)$, to every morphism $S \to T$, a map of sets $Z(T) \to Z(S)$)

 such that the sheaf condition holds: for every Zariski open cover $\{U_i\}_{i\in I}$ of a scheme T and for every $(\varphi_i \in Z(U_i))_{i\in I}$ s.t. φ_i and φ_j map to the same element of $Z(U_i \cap U_j)$ $\forall\, i,j$, there exists unique $\varphi \in Z(T)$ such that $\varphi \mapsto \varphi_i \in Z(U_i)$ $\forall\, i$.

 in other words, the sequence of sets

$$Z(T) \to \prod_i Z(U_i) \rightrightarrows \prod_{i,j} Z(U_i \cap U_j)$$

 is exact

 Examples:
 $*$: $Z(T) = \{*\}$ for all T
 locally constant sheaves
 $\mathcal{O}$: $Z(T) = \Gamma(T, \mathcal{O}_T)$
 the functor of points h_Y: fix a scheme Y, define $Z(T) = \operatorname{Hom}(T, Y)$

 Non-examples:
 $Z(T) = \Gamma(T, \mathcal{K}_T)$ (not even a presheaf)
 $Z(T) =$ two-point set, for all T (violates sheaf axiom for $T = T_1 \amalg T_2$)

Exercise. M_g ($g \geq 2$), $M_g(T) = \left\{ \begin{array}{l} \text{smooth projective morphisms } C \to T \\ \text{geo. fibers all smth conn. cv's of genus } g \end{array} \right\} / \sim$, is *not* a space.

Zariski spaces form a category.

Fact (Yoneda's lemma): the assignment $Y \mapsto h_Y$ determines a fully faithful embedding of the category of schemes into the category of Zariski spaces.

Definition: A Zariski space is said to be **representable** if it is isomorphic to h_Y for some scheme Y.

Examples:
$* \leftrightarrow \operatorname{Spec}\mathbb{Z}$
locally constant sheaf $\leftrightarrow$ disjoint union of copies of $\operatorname{Spec}\mathbb{Z}$
$\mathcal{O} \leftrightarrow \mathbb{A}^1$
$h_Y \leftrightarrow Y$
$(T \mapsto \{\text{rank-one quotients of } \mathcal{O}_T^{\oplus r+1}\}) \leftrightarrow \mathbb{P}^r$ (similarly, Grassmannians)

Exercise. $\Omega : \Omega(T) = \Omega_{T/\operatorname{Spec}\mathbb{Z}}$ is a Zariski space which is not representable.

- Geometry on Zariski spaces

Z a Zariski space

Maps to and from spaces:
a map $T \to Z$ is a morphism $h_T \to Z$, equivalently, an element of $Z(T)$
a map $Z \to S$ is a morphism $Z \to h_S$

Connectedness: we can relate to existence of maps $Z \to \operatorname{Spec}\mathbb{Z} \amalg \operatorname{Spec}\mathbb{Z}$
Separatedness, properness: have valuative criteria
Smoothness, étaleness: have formal criteria
Regular, normal, reduced, ... ?
For schemes, these can be tested locally, e.g. X reduced iff for some equivalently every affine cover $(U_i = \operatorname{Spec} A_i)$, the rings A_i are all reduced (contain no nilpotents)
Have good notion for schemes of Zariski open neighborhoods

Definition: A Zariski space is **algebraic** if there exist schemes U_i and elements $u_i \in Z(U_i)$ such that
(i) $\forall T$, $t \in Z(T)$, the space $S \mapsto \{\,(f,g) \mid f \in \operatorname{Hom}(S,U_i), g \in \operatorname{Hom}(S,T), f^*u_i = g^*t\,\}$ is representable, say, by the scheme T_i (so T_i is the "fiber product" $U_i \times_Z T$),
(ii) $T_i \to T$ is an open inclusion, for each i,
(iii) $\forall T$, $\{T_i\}$ covers T.

Exercise. A Zariski space is algebraic if and only if it is representable.

Notes. Example of $\mathbb{P}^r$ as representable functor, and the final exercise on representability of Zariski spaces, are taken from S. L. Kleiman, "Geometry on Grassmannians and applications to splitting bundles and smoothing cycles," Publ. Math. I.H.E.S. 36 (1969), 281–297; this article attributes the statement on representable spaces to Grothendieck.

The example of the nonrepresentable space Ω was suggested to me by Kai Behrend.