Math 629: Topics in Algebraic Geometry – Algebraic Stacks – Spring 1999 A. Kresch
Lecture 16, March 15, 1999

- Virtual fundamental class: a sample calculation

  V = Calabi-Yau threefold, $c_1(V) = 0$, so for genus 0, exp. dim. $\overline{\mathcal{M}}_{0,0}(V,\beta)$ is 0

  covers of curves lead to positive-dimensional components
  each contributes a virtual number
  Manin (1994) degree d cover of smooth rational curve on V contributes d^{-3}

  example: Q = general smooth quintic in $\mathbb{P}^4$
  (Schubert, 1885) # lines on $Q = 2875$
  (Katz, 1986) # conics $= 609250$, hence

$$\int [\overline{\mathcal{M}}_{0,0}(Q,2)]^{\mathrm{virt}} = 609250 + \frac{2875}{8} = \frac{4876875}{8}$$

  case $d = 2$, explicit computation:
  components of moduli space isomorphic to $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 2)$
  branch locus: $\overline{\mathcal{M}} \to \mathrm{Sym}^2\,\mathbb{P}^1 = \mathbb{P}^2$, étale, admits sections on standard affines U_1, U_2, U_3
  no global section: $\overline{\mathcal{M}}$ is a nontrivial gerbe over $\mathbb{P}^2$

- Interlude: gerbes

  A map of Deligne-Mumford stacks $f\colon V \to X$ is a **gerbe** if f is étale surjective and the relative diagonal $V \to V \times_X V$ is also étale surjective

  Some gerbes, such as $\overline{\mathcal{M}} \to \mathbb{P}^2$, are Zariski locally trivial:
  $X = \bigcup_{i\in I} U_i$, G = an abelian group
  given: G-torsors $B_{ij} \to U_{ij}$, with compatible iso's $B_{ij} \times B_{jk} \to B_{ik}$ on U_{ijk}
  ($BG \times BG \simeq B(G \times G) \to BG$ induces $- \times -$)
  if V is the stackification of the prestack:
  object over T is a triple $(i, f, E \to T)$, $i \in I$, $f\colon T \to X$, $E \to T$ a G-torsor
  morphism: over $h\colon S \to T$ from $(i, g\circ h, E')$ to (j, g, E) is iso $E' \simeq E \times B_{ij}$
  then $V \to X$ is a gerbe, $V \times_X U_i \simeq U_i \times BG$, stabilizer stack $I_V \simeq V \times G$

  More general gerbes are only étale locally trivial
  gerbes $V \to X$ with $I_V \simeq V \times G$ are classified by $H^2(X, G)$
  $X = \mathbb{P}^2_{\mathbb{C}}$, $G = \mathbb{Z}/2$, $H^2(\mathbb{P}^2, G) = \langle c_1(\mathcal{O}(1))\rangle$ (c_1 = bdry. map from $0 \to G \to \mathcal{O}^* \to \mathcal{O}^* \to 0$)

- The computation

  line $\mathbb{P}^1 \subset Q$, rigid: $N = N_{\mathbb{P}^1/Q} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

  universal curve, univ. map:
$$\begin{array}{ccc} C & \xrightarrow{f} & \overline{\mathcal{M}} \times \mathbb{P}^1 \\ {\scriptstyle\pi}\searrow & & \swarrow \\ & \overline{\mathcal{M}} & \end{array}$$

  $[\overline{\mathcal{M}}]^{\mathrm{virt}} = c_2(R^1\pi_* f^* N) = c_1(L)^2$,

  $L = R^1\pi_* f^*\mathcal{O}(-1)$

  duality: $L^\vee \simeq \pi_*\big(\omega_{C/\overline{\mathcal{M}}} \otimes f^*(\mathcal{O}(1))\big)$

Comparison of Pic's: the trivial and nontrivial gerbes over $\mathbb{P}^2$

degree of a line bundle can be defined by restricting to pre-image of a line in $\mathbb{P}^2$, taking first Chern class, and integrating (pushing forward to a point)

$\mathrm{Pic}(\mathbb{P}^2 \times B\mathbb{Z}/2) = \mathbb{Z} \oplus (\mathbb{Z}/2)$

gen. of $\mathbb{Z}$ is $\mathcal{O}(1)$, degree $= 1/2$

$\mathrm{Pic}(\overline{\mathcal{M}}) = \mathbb{Z}$

gen. = square root of $\mathcal{O}(1)$, degree $1/4$

Claim: $L^\vee$ is a generator of $\mathrm{Pic}(\overline{\mathcal{M}})$

Proof: write down a 1-dim'l. family of degree-two maps to $\mathbb{P}^1$

family $y = x^2 + a$, branch locus $\{a, \infty\}$, does *not* extend to complete family

to extend, we must adjoin $b = \sqrt{a}$:

$$\mathbb{P}^1 \to \overline{\mathcal{M}} \qquad \leftrightarrow \qquad \begin{cases} y = x^2 + b^2 \\ c^2 v = u^2 + v^2, \text{ with } c = b^{-1},\ v = y^{-1},\ u = b^{-1}xy^{-1} \end{cases}$$

note $c = 0$ singular fiber

Sections which trivialize $L^\vee$:

on the b-line: $\dfrac{dx}{y}$

on the c-line: $\dfrac{dv}{u}$

transition: $$\frac{dv}{u} = \frac{d(y^{-1})}{b^{-1}xy^{-1}} = -2b\frac{dx}{y}$$

so $L^\vee|_{\mathbb{P}^1}$ has degree 1

$\mathbb{P}^1 \to \overline{\mathcal{M}}$ is of degree 4, so $L^\vee$ has degree $(1/4)$

hence $c_1(L^\vee) = (1/2)[Z]$, where $Z =$ pre-image in $\overline{\mathcal{M}}$ of line in $\mathbb{P}^2$

$c_1(L^\vee)^2 = (1/4)[B\mathbb{Z}/2]$

hence:

$$\int [\overline{\mathcal{M}}]^{\mathrm{virt}} = \int c_1(L)^2 = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$