

- Categorical quotients

Given an algebraic stack X , there may or may not exist an algebraic space M which is universal for maps to algebraic spaces. Such M , if it exists, is called a **categorical quotient**

Example of bad quotient: $X = [\mathbb{A}^n/\mathbb{G}_m] \rightarrow \text{Spec } \mathbb{C}$ collapses non-isomorphic morphisms $\text{Spec } \mathbb{C} \rightarrow X$

Definition: (i) $X \rightarrow M$ is a **uniform categorical quotient** if $X \times_M N \rightarrow N$ is a categorical quotient for any flat map of algebraic spaces $M \rightarrow N$

(ii) $X \rightarrow M$ is a **moduli space** if $X \rightarrow M$ is a uniform categorical quotient, and if $X(\Omega)/\sim \rightarrow M(\Omega)$ is a bijection, for any algebraically closed field Ω .

- Stabilizer $I_X = X \times_{(X \times X)} X \rightarrow X$

- Construction of coarse moduli space

$S =$ locally noetherian base scheme, $X =$ Artin stack, finite type over S

Theorem (Keel-Mori) If X has finite stabilizer, then a moduli space for X exists

Sketch of proof, case X is a Deligne-Mumford stack, groupoid presentation (R, U, s, t, m, e, i) for X
 -introduce notion of *strongly étale morphisms*:

these are étale representable morphisms $X \rightarrow Y$ which induce isomorphism on $\text{Isom}_{\text{Spec } \Omega}(u, u)$
 for all alg. closed fields Ω and all $u: \text{Spec } \Omega \rightarrow X$

alt. description: $(U \times_Y U) \times_{(U \times U)} U \rightarrow U \times_Y U$ factors through R

-show X has a strongly étale cover by quotient stacks $[U_i/G_i]$, $U_i =$ affine scheme, $G_i =$ finite group
 closed point $u \in U$, after $S \rightarrow S'$ étale, we may assume $\text{Isom}(u, u) \simeq G \times u$, some group G

Enumerate $G = \{g_1 = e, g_2, \dots, g_n\}$, define $\mu(j, k)$ by $g_j \cdot g_k = g_{\mu(j,k)}$

let $\tilde{U} = R \times_{s, U, s} R \times_{s, U, s} \dots \times_{s, U, s} R$ ($n - 1$ copies)

let G act on \tilde{U} (on right) by

$$(r_2, \dots, r_n) \cdot g_j = (m(i(r_j), r_{\mu(j,2)}), \dots, m(i(r_j), r_{\mu(j,n)}))$$

let $\tilde{U}^\circ =$ connected component of \tilde{U} containing $\tilde{u} := \text{Isom}(u, u)$

shrink to $\{v \in \tilde{U}^\circ \mid \text{Isom}(v, v) \subset G\text{-orbit of } \tilde{u}\}$ (open by finite stabilizer hypothesis)

shrink to G -invariant affine neighborhood of \tilde{u} , then $[\tilde{U}^\circ/G] \rightarrow X$ is strongly étale

-set $[U_i/G_i] \times_X [U_j/G_j] = [U_{i,j}/G_i]$, then moduli space = $[\coprod_{i,j} U_{i,j}/G_i \rightrightarrows \coprod U_i/G_i]$

- Relation to GIT

Definition: (i) $f: X \rightarrow M$ is a **quotient** if f is bijective on Ω -valued points and is universally submersive ($U \subset M$ open iff $f^{-1}(U)$ open, and same after arbitrary pullback).

(ii) $f: X \rightarrow M$ is a **geometric quotient** if, in addition, $\mathcal{O}_M \rightarrow f_*\mathcal{O}_X$ is an isomorphism.

GIT constructions of moduli spaces (of abelian varieties, of curves, ...) came long before the stack quotient construction, and have the added advantage of guaranteeing (quasi-)projective quotient schemes.

start with: G (a reductive group scheme over S) acting on a finite-type scheme V

additional data of a G -equivariant ample line bundle L on V

identify $V^s \subset V$, and construct V^s/G

under suitable hypothesis, V^s/G is both a geometric quotient and a categorical quotient