

Elementary consequences of coarse moduli space:

$S =$ locally noetherian base scheme, X DM stack, finite type/ S w. finite stabilizer, moduli space $X \xrightarrow{\pi} M$

- Comparison of vector bundles

Assume for all $\text{Spec } \Omega \xrightarrow{u} X$, $\# \text{Isom}(u, u)$ is prime to $\text{char } \Omega$.

Then, a vector bundle F on X is isomorphic to π^*E for some vector bundle E on M if and only if the point stabilizer actions on the fibers of F are all trivial. The vector bundle E , if it exists, is unique up to isomorphism.

- Comparison of cohomology groups

Let $A = \mathbb{Q}$, or let $A = \mathbb{Z}[1/m]$ and assume m is a multiple of all the orders of stabilizer groups of X .

For any sheaf \mathcal{F} of A -modules on X , $R^i\pi_*\mathcal{F} = 0$ for $i > 0$.

Hence $H^i(M, \pi_*\mathcal{F}) \simeq H^i(X, \mathcal{F})$ for all i .

In particular, if \mathcal{E} is a locally constant sheaf of A -modules on M , then $H^i(M, \mathcal{E}) \simeq H^i(X, \pi^*\mathcal{E})$

- Analytic stacks

Take, as base category, (Hausdorff) analytic spaces with the analytic topology (maps are local analytic isomorphisms)

Call a stack analytic if there is a cover by an analytic space and if the diagonal is representable and locally finite

These are analytic Deligne-Mumford stacks.

If X is a smooth proper analytic stack, then cohomology comparison implies Poincaré duality isomorphism $H^{2 \dim X - i}(X, \mathbb{Q}) \simeq H^i(X, \mathbb{Q})^\vee$.

- GAGA

Functor $X \mapsto X^{\text{an}}$

An elementary GAGA result: if X is proper over $\text{Spec } \mathbb{C}$ and \mathcal{F} is a coherent sheaf on X , then $H^i(X, \mathcal{F}) \simeq H^i(X^{\text{an}}, \mathcal{F}^{\text{an}})$.

- Orbifolds

Let us work in the analytic category.

Orbifold data consists of an analytic space M , together with a collection of smooth analytic spaces U_i and finite groups G_i acting effectively, and a covering $\{V_i\}$ of M and analytic isomorphisms $U_i/G_i \simeq V_i$.

Compatibility condition: if $u \in U_i$ and $u' \in U_j$ map to the same point of M , there must be neighborhoods W of u in U_i and W' of u' in U_j , and an analytic isomorphism $W \rightarrow W'$ mapping u to u' and compatible with the projections to M .

An **orbifold** is an equivalence class of orbifold data.

Orbifold data determine an analytic groupoid:

$[R \rightrightarrows U]$

$U = \coprod_i U_i$

$R = \{(u, u', \varphi)\}$, with $u, u' \in U$ mapping to same point of M and $\varphi =$ germ of analytic iso from neighborhood of u to neighborhood of u' , compatible with projections to M

Orbifolds \leftrightarrow smooth analytic stacks with trivial generic stabilizers

The underlying space M does not determine the orbifold structure. If one rules out complex reflections, then M indeed does determine the orbifold structure:

Orbifolds w/o complex reflections \leftrightarrow analytic spaces with quotient singularities

- Gerbes over orbifolds

Let X be a stack: analytic, or algebraic with finite stabilizer

Write $X = [R \rightrightarrows U]$. Let $R' = R/(\text{étale component of stabilizer})$, $Y = [R' \rightrightarrows U]$.

$X \rightarrow Y$ is a gerbe

Y has trivial generic stabilizer

In particular, every smooth stack (analytic, or algebraic with finite stabilizer) is a gerbe over an orbifold

- Which stacks are quotient stacks?

Quotient stacks: $[V/G]$, $V =$ algebraic space, $G =$ algebraic group (all finite type over a field)

Observation: If X is an algebraic orbifold (smooth Deligne-Mumford stack with trivial generic stabilizer) then X is a global quotient.

Which others?

Proposition: let X be a smooth Deligne-Mumford stack of finite type over a field. If X has a finite cover by a Cohen-Macaulay stack, then X is a quotient stack.

Corollary: X as above. If $\dim X \leq 2$ then X is a quotient stack. In general, there exists open $U \subset X$, with U a quotient stack and $\text{codim } X \setminus U \geq 3$.

Open problem: Is every smooth, separated Deligne-Mumford stack, of finite type over a field, a quotient stack?

This is even unknown for gerbes over quasi-projective varieties

Proposition. Let X be a smooth Deligne-Mumford stack of finite type over a field. Assume X has finite stabilizer and the coarse moduli space is a scheme. Let $f: T \rightarrow X$ be a finite cover by a scheme. Then the following are equivalent:

(i) there exists a locally free sheaf \mathcal{E} and epimorphism $\mathcal{E} \rightarrow f_*\mathcal{O}_T$

(ii) every coherent sheaf on X admits a resolution by locally free sheaves

(iii) X is a quotient stack