Math 629: Topics in Algebraic Geometry – Algebraic Stacks – Spring 1999 A. Kresch
Lecture 2, January 18, 1999

- No fine moduli space M_g

Recall: $M_g(T) = \{C \to T\}/\sim$

Why is M_g not a space? Problem: iso between $C_1 \to U_1$ and $C_2 \to U_2$ over $U_1 \cap U_2$ is not uniquely determined.

Example: Let C be a general hyperelliptic curve of genus $g \geq 2$ over $\mathbb{C}$.
Then $\mathrm{Aut}(C) = \{1_C, \sigma\}$ $\quad \sigma =$ hyperelliptic involution.

Let $T' \to T$ be a nontrivial 2-to-1 cover, e.g. $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$, $t \mapsto t^2$.
Then we can construct two non-isomorphic families $C_1 \to T$, $C_2 \to T$, such that fibers over all $t \in T$ are isomorphic to C:
$C_1 = T \times C \to T$
$C_2 = T' \times C/(t,x) \sim (-t, \sigma(x)) \to T$

Suppose there were a scheme M_g such that $\mathrm{Hom}(T, M_g) \leftrightarrow \{\text{families } C \to T\}/\sim$.
$T = \mathrm{Spec}\,\mathbb{C}$: closed pts of $M_g(\mathbb{C})$ correspond to iso. classes of genus g Riemann surfaces.
$\Rightarrow$ both $C_1 \to T$ and $C_2 \to T$ correspond to constant map $T \to \{c\} \subset M_g$.

So we say: "A fine moduli space does not exist."

Remark: we can choose $T' \to T$ to be Zariski locally trivial, and thus get a counterexample to the sheaf axiom for M_g.

Alternative approaches:

(1) sheafify the presheaf M_g – destroys deformation theory analysis

(2) coarse moduli space M_g – a singular scheme which satisfies

$$\{\text{families } C \to T\}/\sim \longrightarrow \{\text{maps } T \to M_g\}$$

not 1-1 in general, but 1-1 when $T = \mathrm{Spec}\,\Omega$, $\Omega =$ alg. closed field.

(3) study

$$\mathcal{M}_g(T) = \text{category} \begin{cases} \text{objects:} & C \to T \text{ smooth families of genus } g \text{ curves} \\ \text{morphisms:} & \begin{matrix} C_1 & \xrightarrow{\sim} & C_2 \\ & \searrow \quad \nearrow & \\ & T & \end{matrix} \quad \text{iso's of families} \end{cases}$$

so, for instance, a line bundle on $\mathcal{M}_g$ consists in the data of, for every family $C \to T$, a line bundle L_C on T; for every Cartesian square $\begin{matrix} B & \to & C \\ \downarrow & & \downarrow \\ S & \xrightarrow{g} & T \end{matrix}$ an isomorphism $L_B \xrightarrow{\sim} g^* L_C$; such that, for any pair of Cartesian squares $\begin{array}{ccccc} A & \to & B & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ R & \xrightarrow{f} & S & \xrightarrow{g} & T \end{array}$, the diagram is commutative:

$$\begin{array}{ccc} L_A & \xrightarrow{\simeq} & f^*L_B \\ \downarrow\sim & & \downarrow\sim \\ (g\circ f)^*L_C & = & f^*g^*L_C \end{array}$$

Definition: $\mathrm{Pic}(\mathcal{M}_g) = \{\text{line bundles on } \mathcal{M}_g\}/\text{isomorphism}$.

- Mumford's modular topologies

Mumford (1963): study $\mathcal{M}_g$ by studying so-called modular families of curves.

No fine moduli space, correspondingly, no universal family of curves.

Next best thing: families $C_i \to T_i$ such that:
- every curve occurs in some family
- in each family, any given curve occurs at most finitely many times
- "$T_i \to \mathcal{M}_g$ étale": formal condition, amounts to saying that the corresponding map $T_i \to M_g$ is branched only over locus of curves with automorphisms; branching is of a specific type (Mumford made an *ad hoc* characterization to handle the case of elliptic curves, and based on this was able to compute Pic).

If we call such families *coverings*, we get a *topology.*
étale topology on schemes (more on this next time)
can form cohomology, and obtain $\mathrm{Pic}(\mathcal{M}_g) \simeq H^1(\mathcal{M}_g, \mathcal{O}^*)$.

- Modular families of elliptic curves

We can consider $\mathcal{M}_{g,n}$, families of genus g curves with n sections (marked points).

$\mathcal{M}_{1,1}$: familes of genus one curves with one section = elliptic curves.

Coarse moduli space: an elliptic curve is characterized up to isomorphism by its j-invariant, $M_{1,1} = \mathbb{A}^1$.
attempt at universal family (over $\mathbb{C}$, say):

$$C_j\colon (y^2 = x^3 + \frac{27}{4}\cdot\frac{1728-j}{j}(x+1))$$

does not extend across $j = 0$, $j = 1728$.
these are precisely the curves with extra automorphisms.
any family containing these curves w/ extra aut's must contain nearby curves multiple times

Exercise. Write down modular families of elliptic curves over $\mathbb{C}$ which contain curves of every j-invariant. Use this to compute $\mathrm{Pic}(\mathcal{M}_{1,1})$. (Hint: can do this with $C_i \to T_i \subset \mathbb{A}^1$, $C_i\colon f_t^{(i)}(x,y) = 0$, $i = 1, 2$, then set $R_{ij} = \{\,(t, t', \varphi) \mid \varphi\colon \text{iso from } (f_t^{(i)}(x,y) = 0) \text{ to } (f_{t'}^{(j)}(x,y) = 0)\,\}$; then $\mathrm{Pic}(\mathcal{M}_{1,1}) = \{\,\coprod_{i,j} R_{ij} \to \mathbb{C}^* \mid \text{cocycle cond.}\,\} \,/\, \{\,\coprod_i U_i \to \mathbb{C}^*\,\}$).

Notes. Mumford's article is "Picard groups of moduli problems," in O. F. G. Schilling, ed., Arithmetical Algebraic Geometry (Purdue University, December 5–7, 1963), Harper & Row, New York, 1965. The overall organization of this lecture, as well as the specific family of elliptic curves mentioned, are taken from that article.