David Harbater lecturing, notes taken by A. Kresch

- Grothendieck's "Esquisse d'un programme"

reference: London Math Society Lecture notes 242

- anabelian conjecture
- dessins d'enfants
- (today's lecture) conjecture that $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ should be determined by its outer actions on $\pi_1(\mathcal{M}_{g,n})$ for all $g, n$

$$G_{\mathbb{Q}} \to \mathrm{Out}(\pi_1(\mathcal{M}_{g,n}))$$

- (Oda, same volume) $\pi_1$ same whether viewed over $\overline{\mathbb{Q}}$ or $\mathbb{C}$
  so can work analytically
  $\pi_1 =$ profinite completion of analytic $\pi_1$

Second way to see outer action

$$1 \to \pi_1(\mathcal{M}_{g,n}/\overline{\mathbb{Q}}) \to \pi_1(\mathcal{M}_{g,n}/\mathbb{Q}) \to G_{\mathbb{Q}} \to 1$$

(it turns out, this is split)

To understand these outer actions: analytically
Easiet case: $g = 0$ ($n \geq 3$) $\mathcal{M}_{g,n}$ is a fine moduli space
triple transitivity of $\mathrm{Aut}(\mathbb{P}^1) \Rightarrow$

$$\mathcal{M}_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3}$$

$n = 3$ point, trivial
$n = 4$ $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

$$\pi_1(\mathcal{M}_{0,4}) = \widehat{F}_2 = \langle x, y, z \mid xyz = 1\rangle$$

outer action of $G_{\mathbb{Q}}$ on $\pi_1$ describes arithmetic of curves over $\mathbb{P}^1$ branched at $0, 1, \infty$
$n > 4$, generalizes $\mathbb{P}^1 \setminus \{0, 1, \infty\}$
$\pi_1(\mathcal{M}_{0,n}) =$ pure mapping class group
$= $ profinite completion of Teichmüller group $\widehat{\Gamma}_0^n$
related to braid group

$g \geq 1$: not a fine moduli space, need stack $\pi_1$
$\mathcal{M}_{g,n}$ is an orbifold, except when $(g, n) = (1, 1)$ or $(2, 0)$

In general,

$$\mathcal{M}_{g,n} = \mathcal{T}/\Gamma_g^n$$

$\mathcal{T} =$ Teichmüller space, a contractible complex manifold

Description 1: given $(g, n)$, pick a topological surface $S$ of type $(g, n)$,
$\Gamma_g^n =$ group of connected components of $\mathrm{Diff}^+(S)$ (+ denotes oriented diffeo's)
$= \mathrm{Diff}^+(S)/\mathrm{Diff}_0^+(S)$ ($\mathrm{Diff}_0^+ =$ conn. component of identity)

$\mathcal{T}_{g,n} = \{\text{hyperbolic structures on } S\}/\operatorname{Diff}+_0(S)$
$\mathcal{M}_{g,n} = \{\text{hyperbolic structures on } S\}/\operatorname{Diff}^+(S)$
Description 2: $\mathcal{M}_{g,n}$ = curves of genus $g$ with $n$ marked points, up to iso
$\mathcal{T}_{g,n}$ = *marked* curves of genus $g$ with ...
here, *marked* means given an iso $S \to X$, up to isotopy
alternatively, for a fixed homology basis on $S$, a specification of the images in $H_*(X)$

We have

$$\dim \mathcal{T}_{g,n} = -3\chi(S) = \dim \mathcal{M}_{g,n}$$

and for $x \in \mathcal{M}_{g,n}$, $\operatorname{Aut}(x)$ is equal to the inertia group of a pre-image of $x$ in $\mathcal{T}_{g,n}$
hence $\pi_1(\mathcal{M}_{g,n}) = \Gamma_g^n$ (analytic $\pi_1$)

- Outer action of $G_{\mathbb{Q}}$ on $\pi_1(\mathcal{M}_{g,n}) = \widehat{\Gamma}_g^n$

simplest case: $(g,n) = (0,4)$, $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0,1,\infty\}$, $\widehat{\Gamma}_{0,4} = \widehat{F}_2$
$G_{\mathbb{Q}}$ outer action: first consider a finite quotient
let $Y \to \mathbb{P}^1$ be a $G$-Galois cover (of curves over $\mathbb{C}$), branched over $0,1,\infty$ (corresp. to surjection $\pi_1 \to G$)
*description*: a triple $(g_1, g_2, g_3)$ s.t. $g_1 g_2 g_3 = 1$, $\langle g_1, g_2, g_3 \rangle = G$, defined up to uniform conjugation
because base points are algebraic, we can get the $G$-Galois cover $Y$ to be defined $/\overline{\mathbb{Q}}$.
hence induced action by $G_{\mathbb{Q}}$
element $\sigma \in G_{\mathbb{Q}}$ transforms $Y \to \mathbb{P}^1$ to $Y^\sigma \to \mathbb{P}^1$, and $(g_1, g_2, g_3)$ to $(h_1, h_2, h_3)$
for each $i$, we know $h_i$ is conjugate to $g_i^{\chi(\sigma)}$, with $\chi(\sigma) \in \widehat{\mathbb{Z}}^*$ the cyclotomic character, characterized by action on $n$-th roots of unity $\zeta_n \overset{\sigma}{\mapsto} \zeta_n^{\chi(\sigma)}$
limit over all curves gives same formalism for $\pi_1$ itself:
$(x, y, z)$ transforms, under $\sigma$, to $(x', y', z')$
$x'$ conjugate to $x^{\chi(\sigma)}$, etc.: individually conjugate, not uniformly conjugate
so, e.g., we may assume $x' = x^{\chi(\sigma)}$

$$\begin{aligned} x &\mapsto x^{\chi(\sigma)} \\ y &\mapsto f^{-1} y^{\chi(\sigma)}) f \\ z &\mapsto g^{-1} z^{\chi(\sigma)}) g \end{aligned}$$

with $f, g \in \widehat{F}_2$.

Belyi: we can modify $f$, without affecting $x \mapsto x^{\chi(\sigma)}$, so that $f$ lies in the commutator subgroup $\widehat{F}_2'$

$$\begin{aligned} x &\mapsto x^{\chi(\sigma)} \\ y &\mapsto f^{-1} y^{\chi(\sigma)}) f, \qquad f \in \widehat{F}_2' \end{aligned}$$

There is a unique lift of outer action to a true action $G_{\mathbb{Q}} \to \operatorname{Aut} \widehat{F}_2$
$f = f_\sigma$ dependins on $\sigma$ and is now specified uniquely
so $\sigma \in G_{\mathbb{Q}}$ determines $(\lambda_\sigma, f_\sigma) \in \widehat{Z}^* \times \widehat{F}_2'$
for *any* $(f, \lambda) \in \widehat{Z}^* \times \widehat{F}_2'$, the pair $(f, \lambda)$ acts on $\widehat{F}_2$ by $x \mapsto x^\lambda$, $y \mapsto f^{-1} y^\lambda f$
yields $G_{\mathbb{Q}} \to \widehat{Z}^* \times \widehat{F}_2'$ (group mapping to semigroup)
if $\mathcal{G}$ denotes group of invertible elements in $Z^* \times \widehat{F}_2'$, then $G_{\mathbb{Q}} \to \widehat{Z}^* \times \widehat{F}_2'$ factors through $\mathcal{G}$
fact: $G_{\mathbb{Q}} \to \mathcal{G}$ is injective (uses result: any curve $/\overline{\mathbb{Q}}$ is a cover of $\mathbb{P}^1$ branched at 3 points)

Question: what is the image of $G_{\mathbb{Q}} \to \mathcal{G}$?
Drinfeld: by ideas from Hopf algebras, get three cycle conditions on elements of image

(I) $f(x,y)f(y,x) = 1$ (2-cycle condition in $\widehat{F}_2 = \widehat{\Gamma}_0^4$)

(II) (3-cycle condition, involving $\chi$, also in $\widehat{\Gamma}_0^4$)

(III) (5-cycle condition in $\widehat{\Gamma}^5 4_0$)

Then he defined Grothendieck-Teichmüller group $\widehat{GT}$ = elements of $\mathcal{G}$ satisfying (I), (II), (III)

Ihara proved that $G_{\mathbb{Q}} \to \mathcal{G}$ factors through $\widehat{GT}$

Recall Grothendieck's suggestion that $G_{\mathbb{Q}}$ should be determined by its outer action on all $\pi_1(\mathcal{M}_{g,n})$

Drinfeld is going further by suggesting enough to consider $g = 0$

"genus zero Teichmüller tower"

raises questions: - is $G_{\mathbb{Q}}$ equal to $\widehat{GT}$?

- does $\widehat{GT}$ encapsulate action on genus 0 Teichmüller tower?

Outer action in terms of $(\lambda, f)$ of $G_{\mathbb{Q}}$ was extended to $(0,n)$ case by Ihara, Matsumoto, Nakamura, i.e., outer action of $\sigma \in G_{\mathbb{Q}}$ on $(\pi_1(\mathcal{M}_{0,n}))$ is determined by knowing $(\lambda_\sigma, f_\sigma)$

Grothendieck: look at $G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{\Gamma}_0^n)$

Drinfeld: look at $G_{\mathbb{Q}} \to \widehat{GT}$

relation between $\widehat{GT}$ and $\mathrm{Out}(\widehat{\Gamma}_0^n)$?

fact: the map $G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{\Gamma}_0^n)$ factors through subgroup $\mathrm{Out}^\#(\widehat{\Gamma}_0^n)$ of elements, invariant under symmetric group action and preserving intertia groups

moreover (L. Schneps and D. H.): $\widehat{GT} = \mathrm{Out}^\#(\widehat{\Gamma}_0^n)$ for $n \geq 5$

so the tower

$$\cdots \to \mathrm{Out}^\#(\widehat{\Gamma}_0^6) \to \mathrm{Out}^\#(\widehat{\Gamma}_0^5) \to \mathrm{Out}^\#(\widehat{\Gamma}_0^4)$$

stabilizes at $n = 5$

- Higher genus

(L. Schneps – H. Nakamura): they found another condition (III$'$) that action of $G_{\mathbb{Q}}$ must satisfy

this condition involves the character $\rho_2$, rather than $\chi$

open question: do (I), (II), (III) $\Rightarrow$ (III$'$)?

or is the picture $G_{\mathbb{Q}} \subset G' \subset \widehat{GT}$

or is there an infinite sequence of intermediate groups $\cdots \subset G'' \subset G' \subset \widehat{GT}$?

- Higher dimensional varieties

Weak version of Grothendieck's conjecture:

Consider all varieties over $\overline{\mathbb{Q}}$ that could be defined over $\mathbb{Q}$

Ihara conjectured $G_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Aut}(\pi_1(\overline{\mathrm{Var}_{\mathbb{Q}}}))$

proved by F. Pop

Recent result (?): enough to consider varieties of dimension $\leq 2$