• Concrete description of stack associated to a groupoid scheme

Given a groupoid scheme (R, U, s, t, m, e, i), we formed a stack $[R \rightrightarrows U]$ over (Sch) (w. étale topology) by considering the associated prestack, and stackifying.

More concrete description:
for a *fixed* étale cover $T' \to T$, there is an associated groupoid:
objects $= \{(\psi\colon T' \to U, \Psi\colon T'' \to R)\}$, with $T'' = T' \times_T T'$
ψ, Ψ must be compatible with structure maps
morphisms $= \{\alpha\colon T' \to R\}$

Exercise. Drawing on your intuition from maps of topological spaces and open coverings, figure out the compatibility conditions (3 of them) on the map α.

The fiber of $[R \rightrightarrows U]$ over T can be thought of as a sort of limit of the groupoids associated to the étale covers $T' \to T$, as T' gets increasingly fine:
object in $[R \rightrightarrows U](T)$ is étale covering $T' \to T$ plus object (ψ, Ψ) in associated groupoid.
morphism from (T'_1, ψ_1, Ψ_1) to (T'_2, ψ_2, Ψ_2) is $\alpha\colon T'_1 \times_T T'_2 \to R$ such that
$s \circ \alpha = \psi_1 \circ \mathrm{pr}_1$,
$t \circ \alpha = \psi_2 \circ \mathrm{pr}_2$,
$m \circ (\alpha \circ \mathrm{pr}_{12}, \Psi_2 \circ \mathrm{pr}_{24}) = m \circ (\Psi_1 \circ \mathrm{pr}_{13}, \alpha \circ \mathrm{pr}_{34})$ as morphisms $T'_1 \times_T T'_2 \times_T T'_1 \times_T T'_2 \to R$.
composition of morphisms:
given (T'_3, ψ_3, Ψ_3) and $\beta\colon T'_2 \times_T T'_3 \to R$, the composite $\beta \circ \alpha$ is the map $T'_1 \times_T T'_3 \to R$ gotten by descent from $\gamma\colon T'_1 \times_T T'_2 \times_T T'_3 \to R$, $\gamma = m \circ (\alpha \circ \mathrm{pr}_{12}, \beta \circ \mathrm{pr}_{23})$.

• Groupoids from stacks

Given a stack, we can try to build a groupoid scheme which allows us to reconstruct the original stack.

Let F be a stack over (Sch). Let U be a scheme. We have:

$$\text{objects } u \in F(U) \leftrightarrow \text{maps } U \to F$$

(any functor can be viewed as a stack; we associate to a scheme U the stack corresponding to its functor of points h_U)

Let u be an object in $F(U)$. The stack $U \times_F U$ is some space R.
leads to a groupoid space (R, U, s, t, m, e, i)
(what is m? T a scheme, $h\colon T \to R$ determines $t_1, t_2\colon T \to U$, with $u(t_1) \xrightarrow{\alpha} u(t_2)$, $k\colon T \to R$ determines $t'_2, t'_3\colon T \to U$, with $u(t'_2) \xrightarrow{\beta} u(t'_3)$; if $t \circ h = s \circ k$ then $t_2 = t'_2$, so get $u(t_1) \xrightarrow{\beta\circ\alpha} u(t'_3)$, hence a third map $m\colon T \to R$)
if R is representable, we get a groupoid scheme, with morphism $[R \rightrightarrows U] \to F$
not generally an equivalence
e.g., $F = Y$ for some scheme Y, can prove equivalence only if $u\colon U \to Y$ is an étale cover,
or more generally a smooth cover ($S \to T$ smth surj. $\Rightarrow \exists\, R \to S$ s.t. $R \to T$ is an étale cover)

• Algebraic stacks

What is desired:
should always get $R = U \times_F U$ representable
should be able to recover $[R \rightrightarrows U] \xrightarrow{\sim} F$ for *suitable* choices U, $u \in F(U)$.

for doing geometry: should be able to define properties of F in terms of those of U, and similarly for morphisms

Digression: descent for properties of schemes and morphisms via étale/smooth/flat covers

Deligne and Mumford chose to study stacks which in a suitable sense are étale locally represented by schemes

Artin: smooth locally

Definition (for now): a stack F is **representable** if F is equivalent to the functor of points of some scheme.

eventually: will disallow some schemes (non-quasi-separated schemes)

will allow representability by some non-scheme spaces, namely algebraic spaces

at that point, will show definition of Deligne-Mumford stack remains unchanged

this alternative definition of representability is needed to obtain the correct definition of Artin stack

Definition (for now): a morphism of stacks $f\colon F \to G$ is **representable** if for any scheme U and object $u \in G(U)$, the fiber product $F \times_G U$ is representable

Proposition: the following are equivalent, for a given stack F

(1) the diagonal $F \to F \times F$ is representable

(2) for any schemes U and V, and morphisms $U \to F$ and $V \to F$, the fiber product $U \times_F V$ is representable

(3) for any scheme U, any morphism $U \to F$ is representable

(4) for any scheme T and objects $t_1, t_2 \in F(T)$, the space $\mathrm{Isom}_{T,t_1,t_2}$ is representable

Proof: we have $\mathrm{Isom}_{T,t_1,t_2} \simeq T \times_{(t_1,t_2),F\times F,\Delta_F} F$, hence (1) $\Leftrightarrow$ (4). There is a natural isomorphism $\mathrm{Isom}_{U\times V,u\circ \mathrm{pr}_1,v\circ \mathrm{pr}_2} \simeq U \times_{u,F,v} V$, so (4) $\Rightarrow$ (2) $\Leftrightarrow$ (3). Finally, for any $(t_1,t_2)\colon T \to F \times F$, we have

$$\begin{aligned} T \times_{F\times F} F &\simeq T \times_{T\times T} T \times T \times_{F\times F} F \\ &\simeq T \times_{T\times T} (T \times_{t_1,F,t_2} T) \end{aligned}$$

which shows (2) $\Rightarrow$ (1).

Definition: a stack F is **algebraic** in the sense of Deligne and Mumford, or is a **Deligne-Mumford stack**, if the diagonal $F \to F \times F$ is representable, quasicompact, and separated, and if there is a scheme U and an étale surjective map $U \to F$.

U as above is called an **atlas**.

How to show a stack is a Deligne-Mumford stack:

first show it is a stack (usually follows from a basic descent result)

then show Isom is representable (may involve a Hilbert scheme argument)

at this point, quasicompactness and separatedness are usually immediate

show there is an étale cover

Shortcut for the last step:

show there is a smooth cover – often easier

show that the diagonal is unramified

and invoke the following result

Proposition: Let F be a stack with representable, separated, quasicompact diagonal, all over a noetherian base scheme S. Suppose there is a scheme X, of finite type over S, and a smooth surjective morphism $X \to F$. If the diagonal of F is unramified, then F is a Deligne-Mumford stack.

Definition: a space which is an algebraic stack in the sense of Deligne and Mumford is called an **algebraic space**.

Algebraic spaces form a category, which contains all quasiseparated schemes as a full subcategory. The category of algebraic spaces sits inside the *2-category* of algebraic stacks.

Definition (final): a stack F is **representable** if F is equivalent to the functor of points of some algebraic space.

Definition (final): a morphism of stacks $f\colon F \to G$ is **representable** if for any quasi-separated scheme U and object $u \in G(U)$, the fiber product $F \times_G U$ is representable.

From now on all schemes will be quasi-separated.

Examples of Deligne-Mumford stacks:
BG, if $G \to S$ is an étale affine group scheme
$[X/G]$, see that this is algebraic directly, or invoke the following result

Proposition: let $F \to G$ be a representable morphism of stacks. If G is algebraic (in the sense of Deligne and Mumford, or in the more general sense of Artin, below), then so is F.

$\mathcal{M}_{1,1}$, stack of elliptic curves
$\mathcal{M}_g$, $\mathcal{M}_{g,n}$, and their compactifications $\overline{\mathcal{M}}_g$, $\overline{\mathcal{M}}_{g,n}$
(see the article of Deligne and Mumford)

Definition: a stack F is **algebraic**, or is an **Artin stack**, if the diagonal is representable, quasicompact, and separated, and if there is a scheme U and a smooth surjective map $U \to F$.

Examples of Artin stacks:
BG, if $G \to S$ is a smooth group scheme
still true for flat group schemes by a result of Artin asserting existence of a smooth atlas
$[X/G]$, will be Deligne-Mumford if G acts with finite, reduced stabilizers
moduli of prestable curves
moduli of vector bundles

Notes. The explicit description of $[R \rightrightarrows U](T)$ is taken from notes written by Fulton during a Spring 1996 seminar course at University of Chicago. The material will appear in the forthcoming book *Introduction to Stacks*, by Kai Behrend, Dan Edidin, Barbara Fantechi, William Fulton, Lothar Göttsche, and Andrew Kresch.