MATH 608, FINAL SOLUTIONS

(1): Given a Cauchy sequence $\{y_n\}_{n\geq 1}$, we need to show that there is a $y_* \in Y$ with $y_n \to y_*$ in the norm topology. We need to deduce this from the assumed completeness of $B(X,Y)$, and $X \neq 0$. The easiest way is to pick $x_0 \neq 0$ in X and a continuous linear functional $f : X \to F$ (underlying field), such that $f(x_0) = 1$. That this is possible follows from Hahn-Banach. Then consider the maps

$$f_n : X \longrightarrow Y$$

given by

$$f_n(x) := f(x)y_n$$

These are bounded due to $|f(x)y_n| \leq C|x||y_n|$ for suitable $C \in \mathbf{R}$. Furthermore, the f_n form a Cauchy sequence, due to

$$||f_n - f_m|| = \sup_{||x||\leq 1} |(f_n - f_m)(x)| \leq C|y_n - y_m|||x|| \longrightarrow 0,\, n,m \to \infty$$

Then by assumption $f_n \longrightarrow f_*$ for a suitable continuous linear functional $f_* : X \longrightarrow Y$. But then $y_n = f_n(x_0) \longrightarrow f_*(x_0)$ as $n \to \infty$, whence Y is complete.

(2) First, the definition

$$||X_0 + a|| := \inf_{y\in X_0} ||y + a||$$

does define a norm. It is well defined since we can put $y = 0$. It satisfies the triangle inequality: given $a, b \in X$ as well as $\epsilon > 0$, pick $y_{1,2} \in X_0$ such that

$$||y_1 + a|| < \inf_{y\in X_0} ||y + a|| + \epsilon,\; ||y_2 + b|| < \inf_{y\in X_0} ||y + b|| + \epsilon$$

Then

$$||y_1 + y_2 + a + b|| \leq ||y_1 + a|| + ||y_2 + b|| < \inf_{y\in X_0} ||y + a|| + \inf_{y\in X_0} ||y + b|| + 2\epsilon$$

Hence

$$\inf_{y\in X_0} ||y + a + b|| < \inf_{y\in X_0} ||y + a|| + \inf_{y\in X_0} ||y + b|| + 2\epsilon$$

and since $\epsilon > 0$ was arbitrary, we are done. Next, assume

$$\inf_{y\in X_0} ||y + a|| = 0$$

This means there exists a sequence $x_n \in X_0$ with $\lim_{n\to\infty} ||x_n + a|| = 0$. By the triangle inequality, x_n is Cauchy, and by completeness, it converges to some $x_0 \in X_0$. But then $a = -x_0 \in X_0$, and so $a + X_0 = 0$ in X/X_0. Next, we show completeness of X/X_0: given a Cauchy sequence $[x_n]$, where we denote by $[x]$ the equivalence class of $x \in X$ in X/X_0, we know by definition that given $\epsilon > 0$, there is $n_0(\epsilon) \in \mathbf{N}$ such that $\forall n, m \geq n_0$, we have $||[x_n] - [x_m]||_{X/X_0} < \epsilon$. It suffices to construct a limit for a subsequence $\{[x_{n_i}]\}_{i\geq 1}$ of $\{x_n\}_{n\geq 1}$. Pick this subsequence such that

$$||[x_{n_{i+1}}]-[x_{n_i}]||_{X/X_0} < 2^{-i-1},\ n_{i+1} > n_i,\ i=1,2,3,\dots$$

By definition, this implies that there exist $a_i \in X_0$ such that

$$||x_{n_{i+1}} - x_{n_i} + a_i||_X < 2^{-i}$$

Now the sequence

$$x_{n_{i+1}} + \sum_{l=1}^{i} a_l = x_{n_1} + \sum_{l=1}^{i}[x_{n_{l+1}} - x_{n_l} + a_l]$$

is Cauchy in X since

$$||\sum_{l=n}^{m}[x_{n_{l+1}} - x_{n_l} + a_l]||_X < 2^{-(n-1)},\ n \leq m$$

Hence by completeness of X it has a limit $x_* \in X$, say. But then $[x_{n_i}] \longrightarrow [x_*]$, since

$$||[x_{n_{i+1}}]-[x_*]||_{X/X_0} \leq ||x_{n_{i+1}} + \sum_{l=1}^{i} a_i - x_*||_X \leq 2^{-i}$$

The reflexivity question is a little trickier. Two approaches emerged in the handed in solutions: one is to exploit an algebraic device, namely the (long) exact sequence

$$0 \longrightarrow X_0 \hookrightarrow X \longrightarrow X/X_0 \longrightarrow 0,$$

where $i: X_0 \hookrightarrow X$ is the standard embedding, while $q: X \longrightarrow X/X_0$ is the standard projection. The 0's added at beginnign and end indicate that the map $X_0 \hookrightarrow X$ is injective while the map $q: X \longrightarrow X/X_0$ is surjective. Also, the sequence is exact, in that the kernel of q in X equals the image of i.

The 2nd device people used was the fact (partially proved in class) that a Banach space is reflexive iff its unit ball is weakly sequentially compact.

Both devices can be used to complete the problem.

(i) First, assume that X is reflexive, and let X_0 be a closed subspace. From what we proved in class (or the 2nd device), we see that X_0 is then reflexive, as well. Next, consider X/X_0. Assuming the 2nd device, we need to show that the unit ball in X/X_0 is weakly sequentially compact. So let $\{[x_n]\}_{n\geq 1}$ with $||[x_n]||_{X/X_0} \leq 1$. By definition, this implies that there is a sequence $\tilde{x}_n \in X$ with $[\tilde{x}_n] = [x_n]$, such that $||\tilde{x}_n||_X < \frac{3}{2}$, say. But then a subsequence $\{\tilde{x}_{n_k}\}_{k\geq 1}$ of $\{\tilde{x}_n\}$ converges weakly in X. Then a fortiori the sequence $\{[\tilde{x}_{n_k}]\}_{k\geq 1}$ converges weakly in X/X_0.

One can also conclude the reflexivity of X/X_0 directly without using the 2nd device, as follows: first, observe that

$$(X/X_0)^* \simeq \{x_* \in X^* | x_*|_{X_0} = 0\}$$

Indeed, each element on the right gives a continuous functional on X/X_0, and conversely, given an element $a \in (X/X_0)^*$, we can define an element $x_* \in X^*$ by $x_*(x) := a(q(x))$, where $q: X \longrightarrow X/X_0$, and then $x_*|_{X_0} = 0$. Also, it is easily seen that under this identification $(X/X_0)^*$ becomes a closed subspace of X^*. Now assume we know the **Claim**: *X is reflexive iff X^* is.* Then $(X/X_0)^*$, being a closed subspace of a reflexive one, is reflexive, and then also (X/X_0) is reflexive. *Proof of Claim*: first, assume X is reflexive, and let $\iota : X \simeq X^{**}$, $\iota_1 : X^* \hookrightarrow X^{***}$ the canonical maps. Also, let $\iota^* : X^{***} \to X^*$ the dual of ι. Then for $x_{***} \in X^{***}$ and arbitrary $x^{**} = \iota(x) \in X^{**}$

$$[\iota_1 \circ \iota^*(x_{***})](x^{**}) = x_{**}(\iota^*(x_{***}) = [\iota(x)](\iota^*(x_{***}) = (\iota^*(x_{***})(x) = x_{***}(\iota(x)) = x_{***}(x_{**})$$

Thus ι_1 is surjective. 2nd, assume X^* is reflexive. Now consider $\iota^* \circ \iota_1 : X^* \longrightarrow X^*$. We show it is the identity, whence ι^* is injective (since ι_1 is an isomorphism by assumption). To see this, compute for $x_* \in X^*$, $x \in X$,

$$(\iota^* \circ \iota_1(x_*))(x) = [\iota_1(x_*)](\iota(x)) = [\iota(x)](x_*) = x_*(x)$$

But if ι^* is injective, then $\iota : X \hookrightarrow X^{**}$ is surjective. Indeed, otherwise $\iota(X)$ would be a proper closed subspace, and by Hahn-Banach we could construct a nonzero element in X^{***} with trivial restriction to $\iota(X)$.

(ii) Now assume that both X_0 and X/X_0 are reflexive. We need to show that so is X. We can use either the exact sequence approach or the one via weak compactness. First, let's use the latter: we need to show that a given sequence $\{x_n\}_{n\geq 1} \subset B_1(0) \subset X$ has a weakly convergent subsequence. Recalling the proof of weak compactness given in class, replace X by $\overline{\mathrm{span}\{x_n\}_{n\geq 1}}$, X_0 by $X_0 \cap \overline{\mathrm{span}\{x_n\}_{n\geq 1}}$ and X/X_0 by $\overline{\mathrm{span}\{x_n\}_{n\geq 1}}/(\overline{\mathrm{span}\{x_n\}_{n\geq 1}} \cap X_0)$, a closed subspace of the former. Thus we may now assume that all spaces $X_0, X, X/X_0$ are separable. Now consider $\{[x_n]\}_{n\geq 1} \subset X/X_0$, and pick a weakly convergent subsequence, which we denote in the same way. Let $[x_0] \in X/X_0$ be its weak limit. We shall now construct an element in X_0^{**} as follows: first, since $X_0 \simeq X_0^{**}$ is separable, so is X_0^* (fact from class). Pick a dense set $\{y_n^*\}_{n\geq 1} \subset X_0^*$, and extend each[1] y_n^* to an element of X^* (Hahn-Banach). Using the standard Cantor diagonal trick (from class), we can find a subsequence of $\{x_n\}_{n\geq 1}$, again denoted as such, such that $\{y_k^*(x_n)\}_{n\geq 1}$ converges for each k. We now claim that the function[2]

$$y_k^* \longrightarrow \lim_{n\to\infty} y_k^*(x_n - x_0)$$

can be extended to all of X_0^*, where it becomes a continuous linear functional, hence an element of X_0^{**}. Indeed, given $x_1^* \in X_0^*$, and $\epsilon > 0$, pick y_k^* with $||x_1^* - y_k^*||_{X_0^*} < \epsilon$, and extend $x_1^* - y_k^*$ to a functional in X^* by Hahn-Banach satisfying the same operator bound, denoted in the same way. Then we have

$$\limsup_{n,m\to\infty} |[(x_1^* - y_k^*) + y_k^*](x_n - x_m)| \leq \epsilon$$

Furthermore, letting $\tilde{x}_1^*$, $\tilde{\tilde{x}}_1^*$ be two extensions of x_1^* to X^*, we have

$$\limsup_{n\to\infty} |(\tilde{x}_1^* - \tilde{\tilde{x}}_1^*)(x_n - x_0)| = 0$$

We conclude that for *any* extension $\tilde{x}_1^*$ of x_1^* to X^* we have

$$\limsup_{n,m\to\infty} |\tilde{x}_1^*(x_n - x_m)| \leq \epsilon,$$

and since $\epsilon > 0$ was arbitrary, the functional (where $\tilde{x}_1^*$ is any extension of x_1^* to an element in X^*)

$$x_1^* \longrightarrow \lim_{n\to\infty} \tilde{x}_1^*(x_n - x_0)$$

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[1] This extension may not be unique

[2] This is a function defined on the set $\{y_k^*\}_{k\geq 1}$

is well-defined[3], bounded and linear, thus defines an element of X_0^{**}. By assumption, it is given by $\iota(x_2)$ for some $x_2 \in X_0$. However, we then have for any $x_* \in X^*$

$$\lim_{n\to\infty} x_*(x_n - x_0) = x_*(x_2),$$

whence $x_0 + x_2$ is the weak limit of $\{x_n\}$.

Next, we show the reflexivity of X using the exact sequences approach. Start with

$$0 \longrightarrow X_0 \longrightarrow X \longrightarrow X/X_0 \longrightarrow 0$$

Here denoting the maps inclusion $i : X_0 \to X$ and projection $q : X \to X/X_0$, we have $\ker(q) = \mathrm{Im}(i)$. We now 'dualize' the long exact sequence, i. e. replace it by

$$0 \longrightarrow (X/X_0)^* \longrightarrow X^* \longrightarrow X_0^* \longrightarrow 0$$

with the maps $q^* : (X/X_0)^* \longrightarrow X^*$, $i^* : X^* \longrightarrow X_0^*$. We claim it is again exact. For example, to see that $\ker(i^*) = \mathrm{Im}(q^*)$, let $x_* \in \ker(i^*)$, which means $x_*|_{X_0} = 0$. Then x_* 'descends' to a well-defined functional $[x]_*$ on X/X_0 via $[x]_*([x]) := x_*(x)$. Linearity and continuity of this are trivially checked. But then $x_* = q^*([x]_*)$, whence $x_* \in \mathrm{Im}(q^*)$. One also easily checks the surjectivity of i^* and the injectivity of q^*. Dualizing once more, we obtain yet another long exact sequence

$$0 \longrightarrow X_0^{**} \longrightarrow X^{**} \longrightarrow (X/X_0)^{**} \longrightarrow 0$$

We can now draw a pretty picture, a 'long exact rectangle' as follows:

$$0 \longrightarrow X_0^{**} \longrightarrow X^{**} \longrightarrow (X/X_0)^{**} \longrightarrow 0$$

$$0 \longrightarrow X_0 \longrightarrow X \longrightarrow X/X_0 \longrightarrow 0$$

Ok, the picture is not so pretty since my latex mastery is suboptimal...there should be vertical arrows as follows: $\iota_{X_0} : X_0 \hookrightarrow X_0^{**}$, $\iota_X : X \hookrightarrow X^{**}$, $\iota_{X/X_0} : X/X_0 \hookrightarrow (X/X_0)^{**}$. First, with all these arrows included, the diagram is commutative. For example, let's check this for the rectangle with vertices X_0^{**}, X^{**}, X_0, X. We need to verify that

$$i^{**} \circ \iota_{X_0} = \iota_X \circ i$$

For any $x_* \in X^*$ and $x_0 \in X_0$, compute

$$(i^{**} \circ \iota_{X_0}(x_0))(x_*) = \iota_{X_0}(x_0)(i^*(x_*)) = (i^*(x_*)(x_0) = x_*(i(x_0)) = (\iota_X \circ i(x_0))(x_*)$$

This implies the desired equality. One checks the remaining commutativity relations analogously.

Now thanks to a mathematical discipline called 'diagram chasing', or more specifically the Five Lemma, one can check that the fact that ι_{X_0}, ι_{X/X_0} are isomorphisms imply that the middle vertical arrow ι_X is also one. We only need to check the surjectivity of ι_X. So let $x_{**} \in X^{**}$ be given. Then

$$q^{**}(x_{**}) = (\iota_{X/X_0} \circ q)(a)$$

for some $a \in X$, and by commutativity we have

$$q^{**}(x_{**} - \iota_X(a)) = 0$$

This implies that there is $b \in X_0$ such that

$$(x_{**} - \iota_X(a) = (i^{**} \circ \iota_{X_0})(b) = (\iota_X \circ i)(b),$$

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[3] in the sense that ti doesn't depend on the choice of extension $\tilde{x}_1^*$

whence

$$x_{**} = \iota_X(a) + (\iota_X \circ i)(b),$$

so ι_X is surjective, as desired.

(3) This is similar to something we did in class. First, let's construct a sequence $\{f_n\}$ on the interval $J_1 := [0,1]$ with the desired property. For this consider for $n \geq 1$ the intervals $I_k^n = [\frac{k-1}{n}, \frac{k}{n}]$, $k = 1, 2, \dots, n$. Also, define $f_k^n := \chi_{I_k^n}$, the corresponding characteristic function. Renumbering these functions as f_i, $i \geq 1$, we have $||f_i||_{L^1} \to 0$ as $i \to \infty$ but for each $x \in [0,1]$ there are infinitely many functions f_i with $f_i(x) = 1$.
To extend this to $\mathbf{R}_{\geq 0}$, cover the latter by the intervals $J_l := [l-1, l]$, $l \geq 1$, and on each of these apply the same construction as for J_1 but with increasingly narrow intervals, for example $I_k^{ln} := l - 1 + [\frac{k-1}{2^l n}, \frac{k}{2^l n}], 1 \leq k \leq 2^l n$. Finally, we need to enumerate all of these countably many functions suitably: for example, enumerating the functions $\chi_{I_k^{ln}}$ for fixed l in some fashion, say as $\{g_n^l\}_{n\geq 1}$, we could let $f_i := g_n^l$ provided $i = 2^l n$, n odd. Finally, to get the desired sequence of functions $\{h_i\}_{i\geq 1}$ on $\mathbf{R}^d$, let $h_i(x) := f_i(|x|)$.

(4) This is a direct consequence of Fubini's theorem: this says that for almost every $y \in [0,1]$, the function $x \longrightarrow |f(x) - f(y)|$ is integrable. Since $|f(y)| < \infty$ by assumption, this implies that $|f(x)|$ is integrable.

(5) First, assume that f satisfies the Lipschitz condition $|f(x) - f(y)| \leq M|x - y|$. Then we show that f is absolutely continuous. Indeed, for $a = a_1 < b_1 < a_2 < b_2 < \ldots < a_k < b_k = b$, we then get

$$\sum_{l=1}^{k} |f(a_l - f(b_l)| \leq \sum_{l=1}^{k} M(b_l - a_l) = M(b-a),$$

whence the supremum over all such decompositions of $[a,b]$ also satisfies this bound. Form class we know that this implies that then

$$f(x) = \int_a^x f'(x)dx$$

for a suitable function $f'(x) \in L^1(\mathbf{R})$. Furthermore, we also know that (Lebesgue differentiation theorem)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

almost everywhere. But then from assumption $|f'(x)| \leq M$ a. e.

Conversely, assume that $f(x)$ is absolutely continuous, and its derivative $f'(x)$ bounded by $|f'(x)| \leq M$ a. e. Then we have

$$f(x) - f(y) = \int_x^y f'(x)dx$$

whence clearly

$$|f(x) - f(y)| \leq \int_x^y |f'(x)|dx \leq M(y-x),$$

as was to be shown.

(6) We shall use Fejer's theorem to show that if for some function $f(x) \in L^2(S^1)$ we have

$$\int_{S^1} f(x)e^{-inx}dx = 0 \ \forall n \in \mathbf{Z},$$

then $f(x) = 0$ a. e. Indeed, Fejer's theorem allows us to approximate $f(x)$ arbitrarily well w. respect to L^2 by a trigonometric polynomial. To wit, given $\epsilon > 0$, pick $g \in C^0(S^1)$ with

$$||f - g||_{L^2(S^1)} < \frac{\epsilon}{2}.$$

Then by Fejer's theorem, we have (uniformly in $x \in S^1$)

$$\lim_{N \to \infty} \frac{S_0(x) + S_1(x) + \ldots + S_N(x)}{N+1} = g(x),$$

where

$$S_l(x) := \sum_{|j| \leq l} \frac{1}{2\pi} \int_{S^1} e^{ij(x-y)} g(y)dy$$

In particular, there is some N_0 such that

$$||g - \frac{S_0(x) + S_1(x) + \ldots + S_{N_0}(x)}{N_0+1}||_{L^2(S^1)} < \frac{\epsilon}{2}$$

But then denoting $\frac{S_0(x)+S_1(x)+\ldots+S_{N_0}(x)}{N_0+1} = \Sigma_{N_0}(x)$, we get

$$||f - \Sigma_{N_0}||_{L^2} < \epsilon$$

Since Σ_{N_0} is a trigonometric polynomial, we have

$$\int_{S^1} f(x)\overline{\Sigma_{N_0}(x)}dx = 0$$

Hence we infer

$$\int_{S^1} |f(x)|^2 dx + \int_{S^1} |\Sigma_{N_0}(x)|^2 dx < \epsilon^2,$$

whence $||f||_{L^2} < \epsilon$. Since $\epsilon > 0$ was arbitrary, we get $||f|_{L^2} = 0$, whence $f(x) = 0$ a. e.