SLOW BLOW UP SOLUTIONS FOR CERTAIN CRITICAL WAVE EQUATIONS.

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1. Introduction

We describe in this article two recent results [11], [12], obtained by the author jointly with W. Schlag and D. Tataru, about singular solutions for the critical wave maps equation, as well as the critical focussing semilinear wave equation. Specifically, the first result [11] establishes for the first time the conjectured formation of singularities for *co-rotational wave maps into the sphere* S^2 *in* $n = 2$ *spatial dimensions*, while the 2nd result [12] establishes the existence of one-point blow-up solutions for the energy critical semilinear equation $\Box u = -u^5$ in $n = 3$ spatial dimensions. Singularity formation for wave maps was previously only known in dimensions $n \geq 3$, while for the semilinear equation above, all previously known blow up solutions become singular along a hyper surface. Furthermore, both results show that there exists a *continuum of blow-up rates* for both problems. This is a new observation in the context of either equation, having previously been observed for a wave maps type equation in $n = 1$ spatial dimension in [19].

1.1. **The semilinear problem.**

Consider the H^1-critical focussing semilinear wave equation in $3 + 1$ spatial dimensions:

$$\Box u = -u^5,\ \Box = -\partial_t^2 + \triangle \tag{1.1}$$

This is a special case of the family of focussing semilinear wave equations (again on $\mathbb{R}^{3+1}$)

$$\Box u = -|u|^{p-1}u,\ p > 1$$

The focussing sign of the nonlinearity suggests the possibility of singularity formation. Indeed, it is straightforward to construct solutions for these which become singular in finite time, even with smooth compactly supported data: let $\alpha = \frac{2}{p-1}$, $C = [\alpha(\alpha+1)]^{\frac{1}{p-1}}$, and

$$u(0, x) = CT^{-\alpha}\phi(x),\ u_t(0, x) = \alpha CT^{-\alpha-1}\phi(x),$$

where $\phi \in C_0^\infty(\mathbb{R}^3)$ satisfies $0 \leq \phi \leq 1$, $\phi(x) = 1$ provided $|x| < 2T$. Then we have

$$u(t, x) = C(T - t)^{-\alpha}$$

for $t < T$ and x in the backward light cone centered at $(T, 0)$, whence the solution blows up at least on the disc $|x| < T$ at time $t = T$.

More sophisticated results, obtained by John [6] and refined by Lindblad [14] and Zhou, showed the existence of singular solutions for suitable initial data of arbitrarily small size, provided $1 < p < 1+\sqrt{2}$, while John (see e. g. [25]) established that for $p > 1 + \sqrt{2}$, small (and smooth) data lead to global solutions. Of particular interest is the energy critical problem corresponding to the case $p = 5$. This is the borderline case in which the problem is still strongly locally well-posed in the energy space H^1. A general criterion ensuring singularity formation for certain (large) initial data was found by Levine [13]: this says that if the energy of the initial data

$$\mathcal{E} := \int_{\mathbb{R}^3} [u_t^2 + |\nabla u|^2 - \frac{1}{6}u^6]dx < 0$$

then a finite time singularity necessarily occurs: more precisely, assuming $(u(0, x), u_t(0, x)) \in H^s \times H^{s-1}$, for some $s \geq 1$, we have

$$||u(t, x)||_{H^1} + ||u_t(t, x)||_{L^2} \to \infty$$

as $t \to T$, for some $T < \infty$. However the argument does not indicate the precise dependence of T on the data except for an upper bound, nor does it provide any idea of the precise blow up dynamics.
Further recent activity related to (1.1) stems from the observation that it admits *static* solutions, of the form

$$W_\lambda(x) := \lambda^{\frac{1}{2}}(1 + \frac{(\lambda|x|)^2}{3})^{-\frac{1}{2}},\ W_1 = W$$

where $\lambda > 0$ is an arbitrary scaling parameter. We note that these solutions have strictly positive energy, whence Levine's argument cannot be directly applied to initial data close to the static ones, $(0, W_\lambda(x)$. Nevertheless, recent work by Kenig-Merle [9] establishes that there exists a *blow-up/global existence dichotomy* for initial data (u_0, u_1) which satisfy the condition $\mathcal{E}(u_0, u_1) < \mathcal{E}(W, 0)$: one gets global existence provided we further have $||\nabla u_0||_{L^2} < ||\nabla W|||_{L^2}$, while one gets finite-time singularity formation provided $||\nabla u_0|| > ||\nabla W||_{L^2}$. However, due to the nature of the arguments, *the precise blow up dynamic of these solutions is unknown*. Indeed, their argument for blow-up relies on a variant of Levine's argument, i. e. a virial type inequality. The following theorem, based on a *purely constructive technique*, provides a precise description of certain blow up solutions whose blow up rate varies along a continuum:

Theorem 1.1. *(K.-Schlag-Tataru 2007)*[12] *Let $\nu > \frac{1}{2}$ and $\delta > 0$. There exists an energy class solution $u(t, x)$ to* (1.1) *which blows up precisely at $x = t = 0$, and which satisfies the following: provided $0 < t < c(\delta)$ for suitable $c(\delta) > 0$ and $0 \leq |x| \leq t$, there is a decomposition*

$$u(t, x) = W_{\lambda(t)}(x) + \eta(t, x),\ \lambda(t) = t^{-1-\nu}$$

with

$$\mathcal{E}_{loc}(\eta(t, .)) := \int_{|x|<t} (\eta_t^2(t, .) + |\nabla\eta|^2(t, .) + |u(t, .)|^6)dx < \delta$$

Furthermore, $(\eta(t, .), \eta_t(t, .)) \in H^{\frac{\nu}{2}+1-} \times H^{\frac{\nu}{2}-}$. Also, we have

$$\sup_{0<t\leq c(\delta)} ||\eta(t, .)||_{H^{1+\mu}} < \infty$$

where $\mu = \frac{\nu}{2(1+\nu)}$.

The interpretation of this result is as follows: the solution $u(t, x)$ decomposes into a *non-oscillatory elliptic part* $W_{\lambda(t)}(x)$ and an *oscillatory radiation part* $\eta(t, x)$. More precisely, as explained below, η itself will consist of a non-oscillatory term and a pure radiation term, the latter being much smaller than the former. The above inequality shows that the radiation term carries very little energy. Indeed, the form of the solution above shows that for t sufficiently small, we have

$$\int_{|x|>t} [u_t^2(t, .) + |\nabla u|^2(t, .) + |u(t, .)|^6)dx < \delta$$

for arbitrary $\delta > 0$. The small-data well-posedness of (1.1) then shows that the solution cannot possibly develop a singularity outside of the backward light cone centered at $(0, 0)$.
The result appears somewhat surprising, as one might surmise that the only possible blow up rate is the one dictated by the ode type blow up, explained above. Indeed, a result by Merle-Zaag shows that this is indeed the case in the conformal range $p \leq 3$.

1.2. **The wave maps equation.** Wave maps $u(t, x) : \mathbb{R}^{n+1} \longrightarrow M$, (M, g) a Riemannian manifold, are critical points of the functional

$$u \longrightarrow \int_{\mathbb{R}^{n+1}} \langle \partial_\alpha u, \partial^\alpha u\rangle_g d\sigma,\ d\sigma = dtdx$$

In local coordinates $\{x_i\}_{i=1,\dots,k}$ on M, this leads to the following system of equations:

$$\Box u^i + \Gamma^i_{jk}\partial_\alpha u^j \partial^\alpha u^k = 0,\ \Box = -\partial_t^2 + \triangle_{\mathbb{R}^n}$$

Here the Γ^i_{jk} are the Riemann-Christoffel symbols of the metric in the corresponding char The problem is *energy super-critical* provided $n \geq 3$, and *energy critical* provided $n = 2$. Singular solutions in the super-critical regime and for suitable targets were constructed[1] by Shatah and Shatah et. al. in [21], [22]. These are of self-similar type, i. e. $u(t,x) = v(\frac{x}{t})$, and obtained by finding such a v as solution of a suitable *harmonic map problem*. This ansatz fails for the critical case, see e. g. [24]. In order to simplify the problem by specialization, introduce the notion of *equivariance*, which is possible provided the target admits an S^1-action. Fix the target $M = S^2$, equipped with its standard metric. Also, let $\rho : S^1 \to \mathrm{Isom}(S^2)$ be an action of S^1 on S^2 by isometries. Then a wave map $u(t,x) : \mathbb{R}^{2+1} \longrightarrow S^2$ is called equivariant, provided we have

$$u(t,\omega x) = \rho(\omega)u(t,x),\ \forall \omega \in S^1$$

One typically chooses $\rho(\omega)$ to be rotation around the z axis by an angle $k\omega$, $k \in \mathbb{Z}$. The case $k = 1$ is called *co-rotational*. In the latter case, one can use the polar angle to completely characterize the wave map. This scalar function only depends on t, r, and calling it (abusing notation) $u(t,r)$, one obtains the scalar wave equation

$$\Box u = \frac{\sin(2u)}{2r^2}, \tag{1.2}$$

with $\Box = -\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r$. The analogy of this problem with (1.1) becomes apparent upon noting that there exist static solutions of the form

$$Q(\lambda r) = 2\arctan(\lambda r),\ \lambda \in \mathbb{R}$$

These correspond to *harmonic maps* from $\mathbb{R}^2$ to S^2, given by stereographic projection.

Problem (1.2) was studied in particular by Struwe [29], who showed that if singularity formation occurs at $(t,r) = (0,0)$, and the initial data are smooth, there exists sequence of times $t_i \to 0$ and numbers λ_i with $\lambda_i t_i \to \infty$, such that the wave map at time t_i decouples into

$$u(t_i, r) = Q(\lambda_i r) + \varepsilon(t_i, r)$$

and we have

$$\lim_{i\to\infty} \int_{r<t} [\varepsilon_t(t_i,r)^2 + \varepsilon_r(t_i,r)^2 + \frac{\sin^2(\varepsilon(t_i,r))}{r^2}] r dr = 0$$

The question remained whether blow up actually occurs, and what the precise rate is at which the λ_i diverge. We have the following

Theorem 1.2. *(K.-Schlag-Tataru 2006)*[11] *Let $\nu > 1$, and denote $\lambda(t) = t^{-1-\nu}$. Then the same conclusions apply as in theorem 1.1, but with*

$$u(t,r) = Q(\lambda(t)r) + \eta(t,r)$$

and $(\eta, \eta_t) \in H^{1+\nu-} \times H^{\nu-}$, and the local energy replaced by

$$\int_{r<t} [\eta_t^2 + \eta_r^2 + \frac{\sin^2(\eta)}{r^2}] r dr$$

The method of proof of this result is similar to the one used to establish theorem 1.1, although there are a number of important structural differences between the equations. We should mention here that in the case of *higher equivariance classes*, more specifically $k \geq 4$, a blow up result was recently established by Rodnianski-Sterbenz [20]. In this case, the equation becomes

$$\Box u = k^2 \frac{\sin(2u)}{2r^2}$$

with static solutions $Q_k(r) = 2\arctan(r^k)$. Rodnianski-Sterbenz show that for suitable initial data satisfying the equivariance condition $u(\omega x) = \omega^k u(x)$, $\omega \in S^1$, which can even be chosen C^∞, the corresponding solution blows up with scaling parameter $\lambda(t) \sim \frac{\sqrt{|\log t|}}{t}$. Moreover, they show that this type of blow up is stable under perturbations (but within this equivariance class). This suggests that also for the co-rotational case,

[1]However, it is a noteworthy open problem to decide whether singular solutions can be constructed for *generic targets*, or if there are geometric obstructions, even in the case $n \geq 3$

stable blow up solutions blow up much faster[2], i. e. closer to the self-similar rate (the latter, however, is precluded by Struwe' s result). A stable blow up regime for (1.2) remains to be found, and we strongly expect (although cannot prove at this point) that the blow up solutions constructed in theorem 1.2 are unstable.

2. Outline of the proof of theorem 1.1

The proof is divided roughly into two parts: first, one constructs an approximate solution but with very high degree of accuracy near the singularity, by treating the equation as an elliptic problem on fixed time slices, i. e. in some sense treating the time derivatives as negligible. More precisely, one needs to distinguish between regions close to the origin and regions near the light cone. We have the following

Proposition 2.1. [12] *For each $k \in \mathbb{N}$, there exists an approximate solution u_k for (1.1) of the form*

$$u_k(t,r) = \lambda^{\frac{1}{2}}(t)[W(R) + \frac{c_k}{(t\lambda)^2}R^2(1+R^2)^{-\frac{1}{2}} + O(\frac{R^2(1+R^2)^{-\frac{3}{2}}}{(t\lambda)^2})],\ R = \lambda(t)r$$

with corresponding error satisfying

$$\Box u_k - u_k^5 = O(\frac{\lambda^{\frac{1}{2}}R}{t^2(t\lambda)^{2k}})$$

Note that the correction to the bulk part $\lambda^{\frac{1}{2}}W(R)$ has local energy of size $O([t\lambda]^{-1})$. This correction flows into the term $\eta(t,r)$ in theorem 1.1, and accounts for the non-oscillatory part of it. The number k will be chosen large later in the 2nd part of the proof in order to ensure the convergence of a certain iteration process.

The proof of the Proposition invokes a finite iteration procedure, alternating between improvements near the spatial origin as well as near the light cone: assuming an approximate solution u_{k-1} with error e_{k-1} has been found, we modify it to $u_k = v_k + u_{k-1}$, thereby replacing e_{k-1} by a smaller error e_k. Of course we let $u_0 = \lambda^{\frac{1}{2}}W(R)$. The intuition now is that near the spatial origin $r = 0$, time derivatives matter less, whence we can replace the linear operator in the equation for v_k by

$$\partial_r^2 + \frac{2}{r}\partial_r + 5u_0^4$$

while near the light cone, the potential term $5u_0^4$ is very small, and we get the linear operator

$$-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r$$

Hence essentially we alternately solve the problems[3]

$$(\partial_r^2 + \frac{2}{r}\partial_r + 5u_0^4)v_{2k+1} \sim e_{2k} \tag{2.1}$$

$$(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)v_{2k} \sim e_{2k-1} \tag{2.2}$$

The first equation easily reduces to

$$(t\lambda)^2\tilde{L}[Rv_{2k+1}] = Rt^2e_{2k},$$

where we have introduced the Sturm-Liouville operator $\tilde{L} = \partial_R^2 + 5W^4(R)$. This is a standard problem solvable via variation of parameters, using the fundamental system

$$\phi(R) = R(1-\frac{R^2}{3})(1+\frac{R^2}{3})^{-\frac{3}{2}},\ \theta(R) = (1+\frac{R^2}{3})^{-\frac{3}{2}}(1-2R^2+\frac{R^4}{9})$$

for $\tilde{L}$.

[2]We follow here the convention, see e. g. [2], of calling rates faster if they are closer to the self-similar rate

[3]The reason for the $\sim$ is that we only remove certain portions of e_k at each step for technical reasons

Remark 2.2. We observe here that the success of our method appears to crucially hinge on the moderate growth of these at $R = \infty$, as well as their regularity at $R = 0$. For example, the analogous procedure for the energy critical Yang-Mills equation under a suitable spherically symmetric ansatz [4]

$$\Box u = \frac{2}{r^2}u(1-u^2),\ u = u(t,r),\ Q(R) = \frac{1-R^2}{1+R^2} \tag{2.3}$$

leads to the fundamental system

$$\phi(R) = \frac{R^{\frac{5}{2}}}{(1+R^2)^2},\ \theta(R) = \frac{-1-8R^2+24R^4\log R+8R^6+R^8}{4R^{\frac{3}{2}}(1+R^2)^2}$$

and our procedure (with $\lambda(t) = t^{-1-\nu}$) fails to produce smaller errors already at the first step. The important difference between (2.3) and (1.1) appears to be that the linearization around the static solution for the latter has a *resonance at the edge of the continuous spectrum*, while for the former, there is an *eigenvalue at zero*. The connection between the spectral properties of the linearization and the convergence of our first approximation step remains to be explained.

The 2nd equation (2.2) above appears to be hyperbolic. Nevertheless, we claim that upon using self-similar coordinates, we can reduce it essentially to a singular ode. Indeed, using $a = \frac{r}{t}$ instead of r, we can write for any function $f(a)$

$$t^2[-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r]f(a) = [(1-a^2)\partial_a^2 + 2(a^{-1}-a)\partial_a]f(a)$$

Of course the right hand side of (2.2) will depend on both t and a, and so we can only approximately treat this problem as an ode in a. Indeed, taking the dependence on t into account, one is led to the following singular operator

$$L_\beta = (1-a^2)\partial_a^2 + 2(a^{-1}+a\beta-a)\partial_a - \beta^2 + \beta$$

where $\beta = (2k-\frac{3}{2})\nu - \frac{1}{2}$. Observe that its fundamental system near $a = 1$ is essentially 1, $(1-a)^{1+\beta}$. The latter function is essentially responsible for the regularity of the solution in theorem 1.1.
The main technical challenge in the proof of theorem 1.1 now is to find suitable function spaces in which one can place v_k, e_k. We refer the reader for the details to the paper [12]. This ends our sketch of the proof of the Proposition.

Having found an approximate solution u_{2k-1} for sufficiently large k, we now try to modify it to an exact solution $u = u_{2k-1} + \varepsilon$. Note that the preceding approximation method will not lead to an exact solution, as the successive time differentiations lead to large constants (at least $k!$ at step k), and hence only formally convergent series. On the other hand, ending the process after finitely many steps and choosing t small enough allows us to control the large constant $k!$ by one copy of t. The idea then is to find ε via construction of a parametrix and Banach iteration in suitable spaces. First, observe that the equation for ε becomes

$$\Box\varepsilon + 5\lambda^2(t)W^4(\lambda(t)x)\varepsilon = N_{2k-1}(\varepsilon) + e_{2k-1}$$

Here N_{2k-1} refers to the nonlinear terms in ε as well as those terms due to the fact that on the left we linearize around u_0 instead of u_{2k-1}. Also, the e_{2k-1} is the error generated by u_{2k-1}. A serious problem here is the time dependence of the operator $\triangle + 5\lambda^2(t)W(\lambda(t)x)$ implicit on the left hand side. By switching to different coordinates, one may remove this time dependence, but at the cost of replacing the time derivative by a dilation type operator: specifically, introduce

$$\tau(t) := \int_t^{t_0}\lambda(s)ds + \nu^{-1}t_0^{-\nu}$$

and the new coordinate $y = \lambda(t)x$. Then writing $\varepsilon(t,x) = v(\tau(t),\lambda(t)x)$, we obtain for v the equation

$$[(\partial_\tau + \dot\lambda\lambda^{-1}y\partial_y)^2 v + \dot\lambda\lambda^{-1}(\partial_\tau + \dot\lambda\lambda^{-1}y\partial_y)v - \triangle v - 5W^4 v] = \lambda^{-2}(\tau)[N_{2k-1}(\varepsilon) + e_{2k-1}]$$

This problem needs to be solved on the τ-interval $[0,\infty]$. We first try to construct a parametrix for the linear operator on the left hand side. Imitating the procedure for the free wave equation, we obtain it by using a

[4] This equation is to be interpreted on $\mathbb{R}^{2+1}$

distorted Fourier transformation associated with the operator $\mathcal{L} := -\partial_R^2 - 5W^4(R)$; this operator appears upon replacing v by $\tilde{\varepsilon} := Rv(\tau, R)$:

$$(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)^2\tilde{\varepsilon} - \dot{\lambda}\lambda^{-1}(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)\tilde{\varepsilon} + \mathcal{L}\tilde{\varepsilon} = \lambda^{-2}(\tau)R[N_{2k-1} + e_{2k-1}] \tag{2.4}$$

We have

Lemma 2.3. [12] *The spectrum of $\mathcal{L}$ consists of $[0,\infty) \cup \xi_d$, where ξ_d is a simple negative eigenvalue. Also, the endpoint of the continuous spectrum is a resonance: we have*

$$\mathcal{L}\phi_0 = 0,\ \phi_0(R) = R(1 - \frac{R^2}{3})(1 + \frac{R^2}{3})^{-\frac{3}{2}}$$

Note that the negative eigenvalue will generally lead to exponential growth at $\tau = \infty$ for the linear evolution. This doesn't cause difficulties, since we can impose the initial data at time $\tau = 0$, i. e. we solve the problem from infinity, imposing zero data at $\tau = \infty$. We also have the following pivotal

Proposition 2.4. [12] *There exists a generalized Fourier basis $\phi(R,\xi)$, $\xi \geq 0$, an eigenstate $\phi_d(R)$ corresponding to ξ_d, and spectral measure $\rho(\xi)d\xi$ with $\rho(\xi) \sim \xi^{-\frac{1}{2}}$ as $\xi \to 0$, $\rho(\xi) \sim \xi^{\frac{1}{2}}$ as $\xi \to \infty$, such that the distorted Fourier transform*

$$\mathcal{F} : f \longrightarrow \widehat{f},\ \widehat{f}(\xi) = \int_0^\infty \phi(R,\xi)f(R)dr,\ \xi \geq 0,\ \widehat{f}(\xi_d) = \int_0^\infty \phi_d(R)f(R)dR$$

is an isometry from $L^2(\mathbb{R}_+)$ to $L^2(\{\xi_d\} \cup \mathbb{R}_+, \rho(\xi)d\xi)$. Furthermore, we have the Fourier inversion formula

$$f(R) = \widehat{f}(\xi_d)\phi_d(R) + \int_0^\infty \phi(R,\xi)\widehat{f}(\xi)\rho(\xi)d\xi$$

With this tool in hand, we can now essentially translate the problem to the distorted Fourier coefficients of the function $\tilde{\varepsilon}$: the difficulty we encounter here has to do with the dilation operator $\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R$, since unlike in the case of the free Laplacian, we do not have the identity $\widehat{\partial_R[Rf]} = -2\xi\partial_\xi\widehat{f}$. Nevertheless, the generalized eigenfunctions approach free eigenfunctions in the regime $\xi R^2 >> 1$, whence one expects the operator

$$\mathcal{K}\widehat{f} := \widehat{R\partial_R f} + 2\xi\partial_\xi\widehat{f}$$

to be bounded in a suitable sense. Fortunately, this turns out to be true, as we have the following

Proposition 2.5. [12] *Introduce the weighted Sobolev type norms*

$$||f||_{L^{2,\alpha}_\rho}^2 := |f(\xi_d)|^2 + \int_0^\infty |f(\xi)|^2\langle\xi\rangle^{2\alpha}\rho(\xi)d\xi$$

Then the operator $\mathcal{K}$ satisfies

$$||\mathcal{K}f||_{L^{2,\alpha}_\rho} \lesssim ||f||_{L^{2,\alpha}_\rho}$$

With this in hand, it is now rather clear how to solve (2.4): use the Fourier representation

$$\tilde{\varepsilon}(\tau,R) = \widehat{\tilde{\varepsilon}}(\tau,\xi_d)\phi_d(\xi) + \int_0^\infty \widehat{\tilde{\varepsilon}}(\tau,\xi_d)\phi(\xi)\rho(\xi)d\xi$$

At the level of the distorted Fourier transform, the problem (2.4) can be formulated as

$$[\partial_\tau^2 + \xi_d]x_d(\tau) = b_d(\tau)$$

$$[(\partial_\tau - 2\dot{\lambda}\lambda^{-1}\xi\partial_\xi)^2 + \xi]x(\tau,\xi) = b(\tau,\xi)$$

Here the functions $b_d(\tau)$, $b(\tau,\xi)$ are essentially the Fourier coefficients of the right hand side of (2.2), but also take into account the error $\mathcal{K}\tilde{\varepsilon}$ incurred upon replacing $\widehat{R\partial_R\tilde{\varepsilon}}$ by $-2\xi\partial_\xi\widehat{\tilde{\varepsilon}}$. The first equation is straightforward to solve upon imposing the condition $x_d(\tau) \to 0$ as $\tau \to \infty$. The 2nd equation, however, requires a somewhat non-standard lemma

Lemma 2.6. [11], [12] *Denote the kernel of the fundamental backward(in time) solution associated with the operator* $(\partial_\tau - 2\dot{\lambda}\lambda^{-1}\xi\partial_\xi)^2 + \xi$ *by* $H(\tau, \xi)$*. Then we have*

$$||H(\tau,\xi)||_{L_\rho^{2,\alpha}\to L_\rho^{2,\alpha+\frac{1}{2}}} \lesssim \tau(\frac{\sigma}{\tau})^C$$

$$||(\partial_\tau - 2\dot{\lambda}\lambda^{-1}\xi\partial_\xi)H(\tau,\xi)||_{L_\rho^{2,\alpha}\to L_\rho^{2,\alpha}} \lesssim (\frac{\sigma}{\tau})^C$$

for a suitable C*.*

The problem now reduces to an iteration in the weighted norms

$$\sup_{\tau\geq 0}\tau^N||x(\tau,\xi)||_{L_\rho^{2,\alpha}} + \sup_{\tau\geq 0}\tau^N||x_d(\tau,\xi)||_{L_\rho^{2,\alpha}}$$

where $\frac{1}{8} \leq \alpha < \frac{\nu}{4}$; the lower bound is necessitated by the multilinear estimates required to bound the nonlinearity, while the upper bound comes from the smoothness of the approximate solution u_{2k-1}. We finally note that the rapid decay of the coefficients (implied by τ^N) is necessary in order to handle the errors generated by $\mathcal{K}\tilde{\varepsilon}$: here one does not gain any powers in τ, and so the *contraction property comes from integrating with respect to* τ. For details, we again refer to the paper [12].

Remark 2.7. The proof of theorem 1.2 is similar to the one described here. At a technical level, the proof in [11] is simpler than the one above, as the spectrum of the linearization around $Q(R) = 2\arctan(R)$ does not contain any negative eigenvalues, and energy conservation together with the equivariance condition immediately preclude the emergence of blow up points away from the origin. The fact that the finite iteration required in the approximation step yields accurate solutions for $\lambda(t) = t^{-1-\nu}$ and arbitrary $\nu > \frac{1}{2}$ for both equations appears quite remarkable.

3. Further problems

We collect here a number of problems which are naturally suggested by our results:

Conjecture 3.1. *The blow up solutions in theorem 1.1 and theorem 1.2 are unstable, but conditionally stable: the latter means that there exists a high-codimensional manifold of initial data which result in the same type of blow up.*

Remark 3.2. We note that Bizon et. al. have observed numerically [2] that there appears to exist a co-dimension one manifold of initial data which separates a global existence regime from a stable blow up regime. Data on the manifold result in blow up of a slower type than in the stable regime. It appears reasonable to suspect that our blow up solutions have data on this manifold.

As the solutions we construct are not C^∞ smooth, we also have

Problem Are there C^∞-smooth data which result in the slow blow up?

Problem Find the stable (within the co-rotational class) blow up rate for co-rotational wave maps.

This already appears quite difficult, and will most likely have to use techniques similar to those employed by Merle-Raphael in the their breakthrough results on the nonlinear Schrodinger equation. Even more difficult is the following

Problem Are there data resulting in stable blow up for critical wave maps even under non-equivariant perturbations? Same question for the semilinear equation without the radiality assumption.

This problem may be well beyond the reach of present technology. Solving it would enable one to construct more exotic blow up solutions: note that the full wave maps system (without symmetry assumptions) and the semilinear equation are invariant under Lorentz transformations. Applying these to one of the solutions guaranteed by theorem 1.2, theorem 1.1, leads to blow up solutions concentrating along a different time like ray. One may make the following

Conjecture 3.3. *There exist blow up solutions for wave maps and the semilinear problem resulting in the bubbling off of several harmonic maps, resp. Aubin Talenti solutions, but along different timelike directions within a backward light cone.*

Another natural question is whether we can generalize these theorems to structurally similar equations. As mentioned earlier, the energy critical Yang-MIlls equation under a spherically symmetric ansatz leads to the problem

$$\Box u = \frac{2}{r^2} u(1-u^2) \tag{3.1}$$

with the static solution $Q(r) = \frac{1-r^2}{1+r^2}$. As observed before, the approximation procedure fails provided one attempts the blow up rate $\lambda(t) = t^{-1-\nu}$. However, it is conceivable that a faster blow up rate closer to the self-similar one allows one to still follow the same process:

Conjecture 3.4. *Problem 3.1 admits blow up solution with scaling parameter $\lambda(t) \sim t^{-1}|\log t|^{\beta}$ for $\beta > 0$ sufficiently large.*

Finally, as already mentioned before, the relation between the spectrum of the linearization around the static solution and the blow up type should be clarified:

Problem Clarify how the spectral properties of the lineatization around the static solution influence the blow up type.

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