NULL-FORM ESTIMATES AND NONLINEAR WAVES

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ABSTRACT. We present bilinear, trilinear as well as quadrilinear null-form estimates arising in connection with the Wave Maps problem. These estimates fit into the program of S. Klainerman instigated in papers such as [4], [1]. While the latter used the framework of $X^{s,b}$-spaces, our estimates involve the Banach spaces introduced by D. Tataru in [13] and further developed by T. Tao in [12]. In this paper we attempt to give a somewhat systematic account of the basic properties of (a certain brand of) these spaces in $n = 3$ space dimensions. In particular, we solve some cases of the 'Division and Summation Problem' for semilinear wave equations whose nonlinearity has null-form structure.

1. INTRODUCTION

A fundamental approach toward the longtime behavior of solutions of semilinear wave equations consists in establishing optimal local well-posedness results. More precisely, for semilinear problems of the form

$$\Box u(\mathbf{x}) = F(u, D_{x,t}u, \dots D^r_{x,t}u)(\mathbf{x}),\ \mathbf{x} := (t,x) \in \mathbf{R}^{n+1}$$

which are invariant under the transformation

$$u(\mathbf{x}) \to \lambda^{\alpha} u(\lambda \mathbf{x}), \tag{1}$$

one aims at proving local well-posedness for initial data in H^s and s as close as possible to the critical Sobolev exponent $s_0 = \frac{n}{2} - \alpha$. Note that $\dot{H}^{s_0}$ is invariant under the above 'rescaling'. In particular, local well-posedness in $\dot{H}^{s_0}$ immediately implies global well-posedness. The work of Klainerman and Machedon in [3], [4], [5], has shown that significant improvements over the well-posedness results based on the classical energy method as well as the 'semiclassical' Strichartz estimates can be achieved by means of so-called null-structures in the nonlinearity[1]. These are special algebraic structures which cause subtle cancellations visible upon working on the Fourier side. For example, the Wave Maps problem in local coordinates leads to an equation of the schematic form

[1] The counterexamples by H. Lindblad in [8] have shown that one generally needs null-structures to improve on the results obtained by means of Strichartz estimates.

$$\Box u = \Gamma(u)\partial_\nu u \partial^\nu u. \tag{2}$$

The expression $Q_0(u,v) = \partial_\nu u \partial^\nu v$ is a fundamental example of a null-form. In [3], Klainerman and Machedon managed to demonstrate almost optimal well-posedness ($s > s_0 = \frac{n}{2}$) for Wave Maps by exploiting this null-form. More precisely, they used so-called $X^{s,\theta}$-spaces, $s > \frac{n}{2}$, $\theta > \frac{1}{2}$, where

$$||u||^2_{\dot{X}^{s,\theta}} = \int_{\mathbf{R}^{n+1}} |\tilde{u}(\tau,\xi)|^2 (1+ |\,|\tau| + |\xi|\,|)^{2s} (1+ |\,|\tau| - |\xi|\,|)^{2\theta} d\tau d\xi.$$

The $X^{s,\theta}$-spaces, where s and θ obey the above constraints, satisfy a list of remarkable properties (see [3]):

(1) **Algebra estimate**: $||uv||_{X^{s,\theta}} \leq C||u||_{X^{s,\theta}}||v||_{X^{s,\theta}}$.

(2) **Null-form estimate**: $||Q_0(u,v)||_{X^{s-1,\theta-1}} \leq C||u||_{X^{s,\theta}}||v||_{X^{s,\theta}}$.

(3) $X^{s,\theta} \subset L^\infty_t H^s$.

(4) $||\chi_T u||_{X^{s,\theta}} \leq CT^\epsilon ||u||_{H^s}$ for *free waves* u, a time cutoff χ_T and a suitable $\epsilon > 0$.

(5) **'Energy inequality':** $||\chi_T u||_{X^{s,\theta}} \leq CT^\epsilon(||\Box u||_{X^{s-1,\theta-1}} + ||u[0]||_{H^s \times H^{s-1}})$ where $u[0] = (u(0), \partial_t u(0))$.

It is not hard to devise an iteration scheme in $X^{s,\theta}$ for the problem (2) based on the preceding properties. Unfortunately, the attempt to prove local well-posedness at the critical level $s_0 = \frac{n}{2}$, $n = 2,3$, by means of a homogeneous analogue of $X^{s,\theta}$ is seen to fail. For example, letting u, v be Schwartz functions and defining

$$||u||_{\dot{X}_k^{\frac{n}{2},\frac{1}{2},p}} = 2^{\frac{kn}{2}} (\sum_{j\in\mathbf{Z}} [2^{\frac{j}{2}} ||Q_j u||_{L^2_t L^2_x}]^p)^{\frac{1}{p}},$$

one has

$$||P_0(uv)||_{\dot{X}_0^{\frac{n}{2},\frac{1}{2},\infty}} \leq C(\sup_{k\in\mathbf{Z}} ||P_k u||_{\dot{X}_k^{\frac{n}{2},\frac{1}{2},1}})(\sup_{k\in\mathbf{Z}} ||P_k v||_{\dot{X}_0^{\frac{n}{2},\frac{1}{2},1}}),$$

but the Besov exponent ∞ on the left-hand side cannot be improved. This 'logarithmic divergence' led Klainerman and Machedon to pose roughly the following

Division Problem: *Find a Banach space X with the same scaling properties as $L^\infty_t \dot{H}^{\frac{n}{2}}$, such that*

(1) *X contains truncated free waves; more precisely, one has an inequality*

$$||\chi_T u||_X \leq C||u||_{\dot{H}^{\frac{n}{2}}}$$

for all free waves u.

(2) *X satisfies the inequality*

$$||\chi_T \Box^{-1} Q_0(u,v)||_X \leq C||u||_X||v||_X.$$

It turns out that the above properties are *too strong*: indeed, simple examples such as the equation $\Box u = Q_0(u,u)$ show that there can't in general be an iteration scheme at the critical level with initial data in $\dot{H}^{\frac{n}{2}}$, basically since the latter doesn't give us control over L^∞. Nevertheless, the Division Problem was solved by D. Tataru [13] if $\dot{H}^{\frac{n}{2}}$ is replaced by $\dot{B}^{\frac{n}{2},1}$, enabling him to demonstrate global well-posedness of Wave Maps in the latter space. In particular, the Division Problem can be solved provided one restricts u, v to frequency ~ 1.

In his quest for establishing global regularity of Wave Maps from $\mathbf{R}^{n+1}$, $n \geq 2$, to S^k, $k \geq 1$, with smooth initial data small in $\dot{H}^{\frac{n}{2}}$, T. Tao [12] solved the following problem in the case of the null-form[2]

$$N(u_1,u_2,u_3) = \sum_{k_i \in \mathbf{Z},\, k_3 > \min\{k_1,k_2\}} P_{k_3}u_1 \partial_\nu P_{k_1}u_2 \partial^\nu P_{k_2}u_3 :$$

Frequency localized Division- and Summation-Problem: *Let the null-form* $N[u_1,u_2,\dots u_r]$, $u_i \in \mathcal{S}(\mathbf{R}^{n+1})$, *be linear in each* u_i; *letting* $(A_\lambda f)(x) := f(\lambda x)$, *assume that it scales according to*

$$N[A_\lambda u_1, A_\lambda u_2, \dots A_\lambda u_r] = \lambda^k A_\lambda(N[u_1,u_2,\dots u_r]).$$

Find a family of Banach spaces $S[m]$, $m \in \mathbf{Z}$, $\mathcal{S}(\mathbf{R}^{3+1}) \subset S[m]$, *which is 'scaling compatible' in the sense that* $2^{-a\frac{k-2}{r-1}}||u(2^a x)||_{S[m+a]} = ||u(x)||_{S[m]}$, $\forall a \in \mathbf{Z}$, *and which satisfies the following properties:*

(1) $P_l S[l] \subset L_t^\infty H^{s_0}$, $s_0 = \frac{n}{2} + \frac{k-2}{r-1}$.

(2) *Truncated free waves at frequency* 2^k *live in* $S[k]$.

[2]This somewhat unpalatable expression, in which the P_k refer to Littlewood-Paley multipliers, arises upon applying an ingenious Gauge Change to the Wave Map u expressed in terms of the ambient coordinates $S^k \hookrightarrow \mathbf{R}^k$. Its crucial feature is that the input with lowest frequency is hit by a derivative.

(3) *Let* $\max_{i=1,\dots r}||P_l u_i||_{S[k]} \leq c_l$*, where* $\{c_l\}$ *is a 'sufficiently flat frequency envelope' in the sense that* $0 < c_a 2^{-\sigma|a-b|} \leq c_b \leq 2^{\sigma|a-b|}c_a$ *for a sufficiently small* $\sigma > 0$[3]*. Then* $\forall l \in \mathbf{Z}$

$$||\chi_T P_l \Box^{-1} N[u_1, u_2, \dots u_r]||_{S[l]} \leq C(\sum_{l\in\mathbf{Z}} c_l^2)^{\frac{r-1}{2}} c_l,$$

where the constants are independent of the smooth cutoff $\chi_T := \chi_0(\frac{t}{T})$*. The operator* $\Box^{-1}$ *refers to the solution of the inhomogeneous wave equation with zero initial data.*

The last property is the reason for the title of the problem: In addition to solving a version of the frequency localized division problem a la Klainerman-Machedon, one has to sum over all possible frequency interactions inside the nonlinearity upon assuming *only the square summability of the different frequency contributions*. Moreover, one has to *recover the original frequency envelope* in the end. The intuition behind this condition is the idea that the 'energy' (more precisely, the norm $\sup_{i=1,\dots r}||u_i||_{L_t^\infty \dot{H}^{s_0}}$, $s_0 = \frac{n}{2} + \frac{k-2}{r-1}$) should not shift significantly amongst frequencies as the nonlinear waves u_i evolve according to a law $\Box u_i = N_i[u_1, u_2, \dots u_r]$, where all N_i satisfy the properties cited above. In particular, solving the *Division and Summation Problem* allows one[4] to prove theorems of the following type:
Let

$$||v_i||_{\dot{H}_x^{s_0}} + ||w_i||_{\dot{H}_x^{s_0-1}} < \epsilon,\ i = 1, \dots r,$$

for suitably small $\epsilon > 0$*,* s_0 *as in the preceding, and also assume that* $u_i(x), v_i(x)$ *are smooth. Then the system*

$$\Box u_i = N_i(u_1, \dots, u_r),\ u_i[0] = (v_i, w_i)$$

admits a globally smooth solution.

Tao's construction of the $S[k]$ is based on so-called null-frame spaces devised by Tataru for his original solution of the Division problem. We shall demonstrate in this paper that Tao's spaces (more precisely, a scaled-down version thereof) are somewhat canonical in that they satisfy additional null-form inequalities arising in the context of the Wave Maps problem for more general targets (see [7]). In particular, we shall solve the *Division and Summation Problem* for $n = 3$ and the null-forms[5]

[3] The σ may depend on the spaces $S[k]$.

[4] Provided the corresponding subcritical well-posedness result is also known.

[5] We denote by R_ν the Riesz multipliers with symbol $\frac{\xi_\nu}{|\xi'|}$, where $\nu = 0, 1, \dots n$, $|\xi'| = (\sum_{i=1}^n |\xi_i|^2)^{\frac{1}{2}}$.

$$N[u_1,u_2,u_3]=\sum_{j=1}^{3}\triangle^{-1}\partial_j(R_ju_1R_\nu u_2-R_\nu u_1R_ju_2)\partial^\nu u_3, \tag{3}$$

$$\begin{aligned}N[u_1,u_2,u_3,u_4]=&\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_\nu u_1R_iu_2-R_iu_1R_\nu u_2)R_ju_3)\partial^\nu u_4\\&-\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_ju_1R_iu_2-R_iu_1R_ju_2)R_\nu u_3)\partial^\nu u_4.\end{aligned} \tag{4}$$

It is shown in [7] that this implies for example the following theorem:

Theorem 1.1. *Let* $\mathbf{H}^2$ *be the hyperbolic plane with standard coordinates* $(\mathbf{x},\mathbf{y})|\,\mathbf{y}>0$. *Also, consider smooth, compactly supported*[6] *initial data* $(\mathbf{x}(0),\,\mathbf{y}(0)):\mathbf{R}^3\to\mathbf{H}^2$, *which are known to result in a smooth Wave Map*

$$(\mathbf{x},\,\mathbf{y}):\mathbf{R}^3\times[-T_0,T_0]\to\mathbf{H}^2$$

on a small time interval $[-T_0,T_0]$, *by means of classical energy estimates. Then there exists a universal constant* $\epsilon>0$ *such that the following holds:*
The inequality

$$\sum_{\alpha=0}^{3}[||\frac{\partial_\alpha\mathbf{x}}{\mathbf{y}}||_{\dot{H}^{\frac{1}{2}}}+||\frac{\partial_\alpha\mathbf{x}}{\mathbf{y}}||_{\dot{H}^{\frac{1}{2}}}]<\epsilon$$

implies that the Wave Map extends smoothly globally in time.

In other words, the estimates derived in this paper enable us to demonstrate *global regularity of Wave Maps* from $\mathbf{R}^{3+1}$ with target $\mathbf{H}^2$ (amongst others), *provided the initial data are small in the critical norm* $\dot{H}^{\frac{3}{2}}$ in the precise sense given above. This settles a conjecture of Klainerman's (see e.g. [2]) in the case of $n=3$ spatial dimensions. The most interesting case $n=2$ has been settled in Tao's work [12] provided the target is a sphere, as well as recently by the author for the case when the target is $\mathbf{H}^2$ (to appear). Also, there is a recent preprint by D. Tataru [14] which claims the case of general target for $n\geq 2$, but proceeding along a different route than in [7]. Critical results of this form have been established in higher spatial dimensions $n\geq 4$ by Tao [12], Klainerman-Rodnianski [6], Shatah-Struwe [10], as well as Nahmod-Stefanov-Uhlenbeck [9]. The problem becomes increasingly difficult in lower spatial dimensions, on account of the increasing scarcity of available a priori estimates.

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[6] In the sense that they are constant outside of a compact set.

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2. Introducing the Banach spaces $S[k]$

We shall now introduce the spaces $S[k]$, $k \in \mathbf{Z}$, which are scaled down versions of Tao's spaces in [12]; these in turn were modelled on Tataru's spaces in [13]. We shall use Tao's terminology for consistency's sake. Recall from the discussion in the previous section that the homogeneous $X^{s,\theta}$-type spaces $\dot{X}_k^{\frac{n}{2}-1,\frac{1}{2},1}$ are too small[7]. On the other hand, the spaces $\dot{X}_k^{\frac{n}{2}-1,\frac{1}{2},\infty}$ would be much too weak. The actual spaces $S[k]$ shall be nested between these two extremes. **From now on, we fix $n=3$.** First a bit of terminology: for every $\omega \in S^2$, we introduce the *null-frame coordinates* (t_ω, x_ω) as follows:

$$t_\omega = (t,x) \cdot \frac{1}{\sqrt{2}}(1,\omega),$$

$$x_\omega = (t,x) - t_\omega \frac{1}{\sqrt{2}}(1,\omega).$$

Following [12], [13], [7], for every small cap $\kappa \subset S^2$, we introduce the *atomic Banach space* $PW[\kappa]$, whose atoms are all Schwartz functions $\phi \in \mathcal{S}(\mathbf{R}^{3+1})$ satisfying

$$||\phi||_{L^2_{t_\omega} L^\infty_{x_\omega}} \leq 1$$

for some $\omega \in \kappa$. These spaces are to be regarded as substitutes for the missing $L^2_t L^\infty_x$-Strichartz estimate. In order to have good product estimates available, we also need a null-frame equivalent for the energy $L^\infty_t L^2_x$: define the space $NFA[\kappa]^*$ as follows:

$$||\phi||_{NFA[\kappa]^*} = \sup_{\omega \notin 2\kappa} ||\phi||_{L^\infty_{t_\omega} L^2_{x_\omega}}.$$

Next, we introduce certain *Fourier multipliers*: the P_k denote standard Littlewood-Paley multipliers localizing to frequency $\sim 2^k$. Also, the multipliers Q_j localize to **modulation** $\sim 2^j$, i. e. they restrict the (space-time) Fourier support to dyadic distance $\sim 2^j$ from the light cone. More specifically, choosing a bump function $m_0(.)$ which is supported around 1, we put

$$\mathcal{F}(Q_j\phi)(\tau,\xi) = m_0(\frac{|\,|\tau| - |\xi|\,|}{2^j})\mathcal{F}\phi.$$

Of course, we shall want to require that $\sum_{j\in\mathbf{Z}} Q_j = 1$ which is possible by choosing

[7]These spaces have the right scaling for our null-forms (3), (4).

$m_0(.)$ appropriately. We shall also use the multipliers $Q_j^{\pm}$, which in addition to the modulation also microlocalize to the upper or lower half space $\tau >< 0$, respectively. Finally, for every cap $\kappa \subset S^2$, we introduce the multipliers $P_{k,\kappa}$, which restrict the Fourier support to frequency ~ 1 and an angular sector of opening $\kappa' \subset \kappa$, where κ' is concentric with κ but of half its size.

We call an operator **disposable** if it is given by convolution with a smooth kernel of bounded L^1-norm. For example, the multipliers P_k, $P_{k,\kappa}$ are disposable, but Q_j is not (its symbol is singular). However, operators of the form $P_k Q_j^{\pm}$, $j \geq k + O(1)$, $P_{k,\kappa} Q_{<k+2l}^{\pm}$, κ of radius 2^l, are disposable, viz. [12].

With the above ingredients, we can define an *auxiliary Banach space* $S[k,\kappa]$ as follows: we let

$$||\phi||_{S[k,\kappa]} := 2^{\frac{k}{2}} ||\phi||_{NFA[\kappa]^*} + |\kappa|^{-\frac{1}{2}} 2^{-\frac{k}{2}} ||\phi||_{PW[\kappa]} + 2^{\frac{k}{2}} ||\phi||_{L_t^\infty L_x^2}.$$

We note that

$$||P_{k,\kappa} Q_{<k}^{\pm} \phi||_{S[k,\pm\kappa]} \leq C ||P_k \phi||_{\dot{X}_k^{\frac{1}{2},\frac{1}{2},1}},$$

which justifies the remarks at the beginning of the section. Also, we have the following **two fundamental bilinear inequalities:**

$$||\phi\psi||_{L_t^2 L_x^2} \leq C \frac{2^{\frac{k'}{2}} |\kappa'|^{\frac{1}{2}}}{\text{dist}(\kappa,\kappa')} ||\phi||_{S[k,\kappa]} ||\psi||_{S[k',\kappa']}, \tag{5}$$

$$||\phi\psi||_{NFA[\kappa]} \leq C \frac{2^{\frac{k'}{2}} |\kappa'|^{\frac{1}{2}}}{\text{dist}(\kappa,\kappa')} ||\phi||_{L_t^2 L_x^2} ||\psi||_{S[k',\kappa']}. \tag{6}$$

The space $NFA[\kappa]$ is of course the dual of $NFA[\kappa]^*$, i. e. it is the atomic Banach space whose atoms are Schwartz functions ϕ satisfying

$$||\phi||_{L_{t_\omega}^1 L_{x_\omega}^2} \leq 1$$

for some $\omega \notin 2\kappa$. This space will be used as a substitute for the customary energy space $L_t^1 L_x^2$.

We can now define the spaces $S[k]$: for every $l < -10$, choose a finitely overlapping collection of caps $\{\kappa\} = K_l$, $\kappa \subset S^2$ of radius 2^l, such that the collection of concentric caps of half that radius covers S^2. Also, define the operators $P_{k,\kappa}$, $\kappa \in K_l$, in such a way that $\sum_{\kappa \in K_l} P_{k,\kappa} = 1$, for every $l < -10$. Finally, we let $\tilde{P}_k$ be a Littlewood-Paley frequency localizer satisfying $\tilde{P}_k P_k = P_k$, and define $\tilde{P}_{k,\kappa}$ accordingly.

Definition:

$$||\phi||_{S[k]} := ||\nabla_{x,t}\phi||_{L_t^\infty \dot{H}^{-\frac{1}{2}}} + ||\nabla_{x,t}\phi||_{\dot{X}_k^{-\frac{1}{2},\frac{1}{2},\infty}} + \sup_{\pm}\sup_{l<-10}(\sum_{\kappa\in K_l} \tilde{P}_{k,\kappa}Q^{\pm}_{<k+2l}\phi||^2_{S[k,\pm\kappa]})^{\frac{1}{2}}.$$

As it stands, this definition is not particularly illuminating; the character of these spaces is only revealed by considering **bilinear expressions**.
Before proceeding, we need to also introduce an (essentially) *dual* family of norms $N[k]$:

Definition: $N[k]$, $k \in \mathbf{Z}$ is the *atomic Banach space* whose atoms are Schwartz functions F of one of the following three types:

(1) $||F||_{L_t^1 L_x^2} \leq 2^{\frac{k}{2}}$.

(2) $||F||_{\dot{X}_k^{-\frac{1}{2},-\frac{1}{2},1}} \leq 1$.

(3) F is microlocalized either to the upper or lower half-space $\tau >< 0$. Also, there is an $l < -10$ such that F lives at frequency $\sim 2^k$ and modulation $< 2^{k+2l}$, and there is a decomposition $F = \sum_{\kappa\in K_l} F_\kappa$ with the property that

(a) F_κ has Fourier support in the angular sector with opening $\kappa' \subset \kappa$ where κ' is concentric with κ and of half its radius.

(b)
$$(\sum_{\kappa\in K_l} ||F_\kappa||^2_{NFA[\pm\kappa]})^{\frac{1}{2}} \leq 2^{\frac{k}{2}},$$
where the sign is chosen according to whether F is microlocalized to $\tau > 0$ or $\tau < 0$.

We shall place the k-th frequency component of the null-forms into $N[k]$. This is justified by means of the

'Energy inequality': Introduce the notation

$$||\phi||_{S[k]([-T,T]\times\mathbf{R}^3)} := \inf_{\psi\in\mathcal{S}(\mathbf{R}^{3+1}),\, \psi|_{[-T,T]}=\phi|_{[-T,T]}} ||\psi||_{S[k]},$$

and similarly for $N[k]([-T,T]\times\mathbf{R}^{3+1})$. Then

$$||P_k\phi||_{S[k]([-T,T]\times\mathbf{R}^{3+1})} \leq C(||P_k\Box\phi||_{N[k]([-T,T]\times\mathbf{R}^{3+1})} + ||P_k\phi[0]||_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}}).$$

The proof of (essentially) this is given in [12].
Before proceeding with the properties of these spaces, we need to introduce some tools:

Classical and Improved Bernstein's inequality: For any measurable set $R \subset \mathbf{R}^n$, we have

$$||\mathcal{F}^{-1}(\chi_R \mathcal{F} f)||_{L^p_x} \leq C|R|^{\frac{1}{2}-\frac{1}{p}}||f||_{L^2_x}. \tag{7}$$

Also, for $n = 3$ and $f \in \mathcal{S}(\mathbf{R}^{3+1})$, we have the inequality

$$||P_k Q_j f||_{L^2_t L^\infty_x} \leq C_\epsilon 2^{\min\{\frac{j-k}{2+\epsilon},0\}}||P_k Q_j f||_{L^2_t L^2_x}, \tag{8}$$

for every $\epsilon > 0$. The proof of the latter assertion is found in [12] [8].

Terminology: The Riesz operators R_ν, $\nu = 0, 1, 2, 3$, refer to the operators $\partial_\nu(\sqrt{-\triangle_x})^{-1}$. We usually omit the subscript for the operators $\triangle_x$, ∇_x, understanding that they only refer to the space variables. The symbol ∇^{-1} is a shorthand for $\sqrt{-\triangle}^{-1}$. When we consider an expression of the form $P_0(AB[CD])$, we shall refer to A, B, C, D, as **inputs**, and the whole expression as **output**. Also, when referring to [,], we mean $[CD]$, while (,) would refer to $(AB[CD])$: the shape of brackets matters in the discussion.

We now state a fundamental lemma which is the homogeneous (and scaled-down) analogue of the property $X^{s-1,\theta-1} \times X^{s,\theta} \subset X^{s-1,\theta-1}$ in [3]: It is essentially[9] due to Tao in [12].

Lemma 2.1. *Let $j \leq \min\{k_1, k_2\} + O(1)$. Also, let F, ψ be Schwartz functions, the former at frequency $\sim 2^{k_1}$ and modulation $\sim 2^j$, the latter at frequency $\sim 2^{k_2}$. Then the following inequalities hold for suitable δ_1, $\delta_2 > 0$:*

$$||P_k(F\psi)||_{N[k]} \leq C 2^{-\delta_1|k-\max\{k_1,k_2\}|} 2^{-\delta_2|j-\min\{k_1,k_2\}|}||F||_{\dot{X}^{\frac{1}{2},-\frac{1}{2},\infty}_{k_1}}||\psi||_{S[k_2]},$$

$$||\nabla_x P_k(F\psi)||_{N[k]} \leq C 2^{-\delta_1|k-\max\{k_1,k_2\}|} 2^{-\delta_2|j-\min\{k_1,k_2\}|}||F||_{\dot{X}^{\frac{1}{2},-\frac{1}{2},\infty}_{k_1}}||\nabla_x \psi||_{S[k_2]}.$$

[8]In the case $n = 2$; the case $n = 3$ is proved similarly. Note that this inequality is really a version of the customary Strichartz inequality. The condition $\epsilon > 0$ is a reflection of the failure of the endpoint Strichartz estimate in $n = 3$ dimensions.

[9]We only have to modify the case of high-high interactions.

Remark: The 2nd inequality is contained in [12]. The first is weaker for high-low interactions but stronger for high-high interactions.

Proof : We only prove the first inequality for **high-high interactions**. We may rescale to $k_1 = k_2 + O(1) = 0$, whence $k < O(1)$. We always let C denote a large positive constant.

(1): Estimate $P_k(FQ_{\geq j-C}\psi)$: we split into two cases:

(1a): $j < 100k$:

$$2^{-\frac{k}{2}}||P_k(FQ_{\geq j-C}\psi)||_{L_t^1 L_x^2} \leq C2^{-\frac{k}{2}}||F||_{L_t^2 L_x^\infty}||Q_{\geq j-C}\psi||_{L_t^2 L_x^2}$$
$$\leq C2^{-\frac{j}{2}}2^{\delta j}2^{-\frac{k}{2}}||F||_{L_t^2 L_x^2}||\psi||_{S[k_2]},$$

where we have employed (8) as well as (7). Our assumption implies that this estimate verifies the lemma.

(1b): $j \geq 100k$: Use (7):

$$2^{-\frac{k}{2}}||P_k(FQ_{\geq j-C}\psi)||_{L_t^1 L_x^2} \leq C2^{-\frac{k}{2}}2^{\frac{3k}{2}}||P_k(FQ_{\geq j-C}\psi)||_{L_t^1 L_x^1}$$
$$\leq C2^k 2^{-\frac{j}{2}}||F||_{L_t^2 L_x^2}||\psi||_{S[k_2]}.$$

This is again acceptable.

(2): The estimate for $P_k Q_{\geq j-C}(FQ_{<j-C}\psi)$:

(2a): $j < 100k$:

$$||P_k Q_{\geq j-C}(FQ_{<j-C}\psi)||_{\dot{X}_k^{-\frac{1}{2},-\frac{1}{2},1}} \leq C2^{-\frac{j}{2}}2^{-\frac{k}{2}}||F||_{L_t^2 L_x^\infty}||Q_{<j-C}\psi||_{L_t^\infty L_x^2}$$
$$\leq C2^{-\frac{j}{2}}2^{-\frac{k}{2}}2^{\delta j}||F||_{L_t^2 L_x^2}||\psi||_{S[k_2]}.$$

(2b): $j \geq 100k$: use Bernstein's inequality similarly to case (1a).

(3): The estimate for $P_k Q_{<j-C}(FQ_{<j-C}\psi)$. Note that $j \leq k + O(1)$. Observe that we can write

$$P_k Q_{<j-C}(FQ_{<j-C}\psi) = \sum_{\pm,\pm,\pm} P_k Q^{\pm}_{<j-C}(Q^{\pm}_j F Q^{\pm}_{<j-C}\psi)$$
$$= \sum_{\pm,\pm,\pm} \sum_{\kappa_{1,2}\in K_{\frac{j+k}{2}-10}, \mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j+k}{2}}} P_k Q^{\pm}_{<j-C}(P_{k_1,\kappa_1} Q^{\pm}_j F P_{k_2,\kappa_2} Q^{\pm}_{<j-C}\psi).$$

By the triangle inequality, we have

$$||P_k Q_{<j-C}(FQ_{<j-C}\psi)||_{N[k]}$$
$$\leq \sum_{\pm,\pm,\pm} \sum_{\kappa_{1,2}\in K_{\frac{j+k}{2}-10}, \mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j+k}{2}}} ||P_k Q^{\pm}_{<j-C}(P_{k_1,\kappa_1} Q^{\pm}_j F P_{k_2,\kappa_2} Q^{\pm}_{<j-C}\psi)||_{N[k]}.$$

On the other hand, observe that

$$P_k Q^{\pm}_{<j-C}(P_{k_1,\kappa_1} Q^{\pm}_j F P_{k_2,\kappa_2} Q^{\pm}_{<j-C}\psi)$$
$$= \sum_{\kappa\in K_{\frac{j-k}{2}-10}, \mathrm{dist}(\pm\kappa,\pm\kappa_1)\sim 2^{\frac{j-k}{2}}} P_{k,\kappa} Q^{\pm}_{<j-C}(P_{k_1,\kappa_1} Q^{\pm}_j F P_{k_2,\kappa_2} Q^{\pm}_{<j-C}\psi).$$

Therefore (6) implies that

$$2^{\frac{k}{2}}||P_k Q^{\pm}_{<j-C}(P_{k_1,\kappa_1} Q^{\pm}_j F P_{k_2,\kappa_2} Q^{\pm}_{<j-C}\psi)||_{N[k]}$$
$$\leq \Big(\sum_{\kappa\in K_{\frac{j-k}{2}-10}, \mathrm{dist}(\pm\kappa,\pm\kappa_1)\sim 2^{\frac{j-k}{2}}} ||P_{k,\kappa} Q^{\pm}_{<j-C}(P_{k_1,\kappa_1} Q^{\pm}_j F P_{k_2,\kappa_2} Q^{\pm}_{<j-C}\psi)||^2_{NFA[\pm\kappa]}\Big)^{\frac{1}{2}}$$
$$\leq C\frac{2^{\frac{j+k}{2}}}{2^{\frac{j-k}{2}}}||P_{k_1,\kappa_1} Q^{\pm}_j F||_{L^2_t L^2_x}||P_{k_2,\kappa_2} Q^{\pm}_{<j-C}\psi||_{S[k_2,\pm\kappa_2]}.$$

We have used the disposability of the operator $P_{k,\kappa} Q^{\pm}_{<j-C}$, viz. the comments in the preceding section.
Using the Cauchy-Schwarz inequality as well as Plancherel's theorem, we conclude that

$$||P_k Q_{<j-C}(FQ_{<j-C}\psi)||_{N[k]}$$
$$\leq C 2^{\frac{j}{2}} 2^{\frac{k}{2}} [2^{-\frac{j}{2}}||P_{k_1}F||_{L^2_t L^2_x}] \sup_{\pm}\Big(\sum_{\kappa\in K_{\frac{j+k}{2}-10}} ||P_{k_2,\kappa} Q^{\pm}_{<j-C}\psi||^2_{S[k_2,\pm\kappa]}\Big)^{\frac{1}{2}}.$$

The desired inequality follows upon observing that

$$\sup_{\pm}\Big(\sum_{\kappa\in K_{\frac{j+k}{2}-10}} ||P_{k_2,\kappa} Q^{\pm}_{<j-C}\psi||^2_{S[k_2,\pm\kappa]}\Big)^{\frac{1}{2}} \leq C|k|||\psi||_{S[k_2]}.$$

■

The preceding lemma entails the following consequence (simply copy the proof in [12] using the above lemma):

Lemma 2.2. *Let F, ψ be Schwartz functions. Then the following inequalities hold:*

(1) *Let $k_1 = k_2 + O(1)$. Then*

$$||P_k(P_{k_1} F P_{k_2}\psi)||_{N[k]} \leq C 2^{\delta(k-k_1)} ||P_{k_1}\nabla_x F||_{N[k_1]} ||P_{k_2}\psi||_{S[k_2]}$$

for suitable $\delta > 0$.

(2)
$$||P_k \nabla_x(\phi P_{k_1} F)||_{N[k]} \leq C(||\phi||_{L_t^\infty L_x^\infty} + \sup_k ||P_k \nabla_x \phi||_{S[k]}) ||\nabla_x P_{k_1} F||_{N[k_1]}.$$

3. Bilinear estimates

The estimates in this section should be compared to the bilinear null-form estimates contained in [1]. The latter were proved for free waves, which corresponds to setting $S[k] = \dot{X}_k^{\frac{1}{2},\frac{1}{2},1}$ in our context. The fact that our spaces $S[k]$ are larger makes our estimates correspondingly weaker. First, we state the following basic

Theorem 3.1. *Let ϕ_1, ϕ_2 be Schwartz functions. Then the following inequalities hold true for $\frac{1}{2} \geq \epsilon > 0$ and $\epsilon > \delta > 0$:*

$$||P_k Q_j(P_{k_1}\phi_1 P_{k_2}\phi_2)||_{\dot{X}_k^{-\epsilon,\epsilon,\infty}} \leq C_{\epsilon,\delta} 2^{\delta \min\{j-\min\{k_1,k_2,k\},0\}} 2^{-\frac{|k_1-k_2|}{2}} \prod_{i=1,2} ||P_{k_i}\phi_i||_{S[k_i]},$$

$$||P_k Q_j(P_{k_1}\phi_1 P_{k_2}\phi_2)||_{\dot{X}_k^{-\frac{1}{2},\frac{1}{2},\infty}} \leq C_\epsilon 2^{\frac{1}{2+\epsilon}\min\{j-\min\{k_1,k_2,k\},0\}} 2^{-|k_1-k_2|} \prod_{i=1,2} ||P_{k_i}\phi_i||_{S[k_i]}.$$

Also, the following inequality holds true for every $\epsilon > 0$:

$$||P_k(P_{k_1}\phi_1 P_{k_2}\phi_2)||_{L_t^2 L_x^{2+\epsilon}} \leq C_\epsilon 2^{k\frac{\epsilon}{4+2\epsilon}} 2^{-\frac{|k_1-k_2|}{2}} \prod_{i=1,2} ||P_{k_i}\phi_i||_{S[k_i]}.$$

In particular, we have the inequality

$$||P_0\phi||_{L_t^4 L_x^q} \leq C_q ||P_0\phi||_{S[0]}$$

for every $q > 4$. Interpolating with $L_t^\infty L_x^2$ gives

$$||P_0\phi||_{L_t^p L_x^q} \leq C_{p,q} ||P_0\phi||_{S[0]},$$

provided $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$, $p\geq 4$.

Proof : The 2nd and third inequality follow in the same way as the first (using the improved Bernstein's inequality (8) for the third). As for the first inequality, we distinguish between different cases:

(1) *High-High interactions*: $k_1=k_2+O(1)>k+5$.

We decompose

$$P_k(P_{k_1}\phi_1 P_{k_2}\phi_2)=P_kQ_{<k+10}(P_{k_1}\phi_1 P_{k_2}\phi_2)+P_kQ_{\geq k+10}(P_{k_1}\phi_1 P_{k_2}\phi_2). \tag{9}$$

We commence with the 2nd term on the right-hand side. Freeze its modulation to size $\sim 2^j$, $j\geq k+10$:

$$\begin{aligned}&P_kQ_j(P_{k_1}\phi_1 P_{k_2}\phi_2)=P_kQ_j(P_{k_1}\phi_1 P_{k_2}Q_{\geq j-10}\phi_2)\\&+P_kQ_j(P_{k_1}Q_{\geq j-10}\phi_1 P_{k_2}Q_{<j-10}\phi_2)\\&+\sum_{\pm}P_kQ_j(P_{k_1}Q^{\pm}_{<j-10}\phi_1 P_{k_2}Q^{\pm}_{<j-10}\phi_2),\end{aligned} \tag{10}$$

where the $\pm$-signs in the last term all match. Now we estimate

$$\begin{aligned}&||P_kQ_j(P_{k_1}Q_{\geq j-10}\phi_1 P_{k_2}\phi_2)||_{L^2_tL^2_x}\leq C2^{\frac{3k}{2}}||P_{k_1}Q_{\geq j-10}\phi_1||_{L^2_tL^2_x}||P_{k_2}\phi_2||_{L^\infty_tL^2_x}\\&\leq C2^{\frac{3k-j-2k_1}{2}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.\end{aligned}$$

From this the claim of the theorem easily follows. For the third summand on the right-hand side of (10), note that it vanishes unless $j=k_1+O(1)$:

$$\begin{aligned}&P_kQ^{\pm}_j(P_{k_1}Q^{\pm}_{<j-10}\phi_1 P_{k_2}Q^{\pm}_{<j-10}\phi_2)\\&=\sum_{\kappa_{1,2}\in K_{k-k_1-10},\,\mathrm{dist}(\kappa_1,-\kappa_2)\sim 2^{k-k_1}}P_kQ^{\pm}_{k_1+O(1)}(P_{k_1,\kappa_1}Q^{\pm}_{<j-10}\phi_1 P_{k_2,\kappa_2}Q^{\pm}_{<j-10}\phi_2)\end{aligned}$$

Now (5) yields the following:

$$\begin{aligned}&||\sum_{\kappa_{1,2}\in K_{k-k_1-10},\,\mathrm{dist}(\kappa_1,-\kappa_2)\sim 2^{k-k_1}}P_kQ^{\pm}_j(P_{k_1,\kappa_1}Q^{\pm}_{<j-10}\phi_1 P_{k_2,\kappa_2}Q^{\pm}_{<j-10}\phi_2)||_{L^2_tL^2_x}\\&\leq\sum_{\kappa_{1,2}\in K_{k-k_1-10},\,\mathrm{dist}(\kappa_1,-\kappa_2)\sim 2^{k-k_1}}||P_kQ^{\pm}_j(P_{k_1,\kappa_1}Q^{\pm}_{<j-10}\phi_1 P_{k_2,\kappa_2}Q^{\pm}_{<j-10}\phi_2)||_{L^2_tL^2_x}\\&\leq C2^{k-k_1}|k-k_1|^2||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.\end{aligned}$$

The claim of the theorem again follows. We proceed to the first term of (9). We commence by reducing the inputs to modulation $< 2^{2k-k_1}$:

$$\begin{aligned}&P_kQ_{<k+10}(P_{k_1}\phi_1P_{k_2}\phi_2)=P_kQ_{<k+10}(P_{k_1}Q_{\geq 2k-k_1}\phi_1P_{k_2}\phi_2)\\&+P_kQ_{<k+10}(P_{k_1}Q_{<2k-k_1}\phi_1P_{k_2}Q_{\geq 2k-k_2}\phi_2)\\&+P_kQ_{<k+10}(P_{k_1}Q_{<2k-k_1}\phi_1P_{k_2}Q_{<2k-k_2}\phi_2).\end{aligned}$$

The first two terms are again easy to estimate

$$\begin{aligned}&||P_kQ_{<k+10}(P_{k_1}Q_{\geq 2k-k_1}\phi_1P_{k_2}\phi_2)||_{L_t^2L_x^2}\leq C2^{\frac{3k}{2}}||P_{k_1}Q_{\geq 2k-k_1}\phi_1||_{L_t^2L_x^2}||P_{k_2}\phi_2||_{L_t^\infty L_x^2}\\&\leq C2^{\frac{k-k_1}{2}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.\end{aligned}$$

As to the last term, we freeze its modulation to size $\sim 2^j$, $j<k+10$, and decompose it further:

$$\begin{aligned}&P_kQ_j(P_{k_1}Q_{<2k-k_1}\phi_1P_{k_2}Q_{<2k-k_2}\phi_2)\\&=P_kQ_j(P_{k_1}Q_{<\min\{2k-k_1,j-C\}}\phi_1P_{k_2}Q_{<\min\{2k-k_2,j-C\}}\phi_2)\\&+P_kQ_j(P_{k_1}Q_{j-C\leq .<2k-k_1}\phi_1P_{k_2}Q_{<2k-k_2}\phi_2)\\&+P_kQ_j(P_{k_1}Q_{<\min\{2k-k_1,j-C\}}\phi_1P_{k_2}Q_{j-C\leq .<2k-k_2}\phi_2).\end{aligned}\tag{11}$$

From elementary geometry, we have

$$\begin{aligned}&P_kQ_j(P_{k_1}Q_{<\min\{2k-k_1,j-C\}}\phi_1P_{k_2}Q_{<\min\{2k-k_2,j-C\}}\phi_2)\\&=\sum_{\pm,\pm}\sum_{\kappa_{1,2}\in K_{\frac{j+k}{2}-k_1-10},\,\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j+k}{2}-k_1}}\\&P_kQ_j(P_{k_1,\kappa_1}Q^{\pm}_{<\min\{2k-k_1,j-C\}}\phi_1P_{k_2,\kappa_2}Q^{\pm}_{<\min\{2k-k_2,j-C\}}\phi_2).\end{aligned}$$

Now we use (5), as well as the Cauchy-Schwarz inequality and the easily verified

$$\begin{aligned}&\Big(\sum_{\kappa\in K_{\frac{j+k}{2}-k_1-10}}||P_{k_1,\kappa}Q^{\pm}_{<\min\{j-C,2k-k_1\}}\phi_1||^2_{S[k_1,\pm\kappa]}\Big)^{\frac{1}{2}}\\&\qquad\qquad\leq C|j-k|||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.\end{aligned}$$

Thus

$$\begin{aligned}&||P_kQ_j(P_{k_1}Q_{<\min\{2k-k_1,j-C\}}\phi_1P_{k_2}Q_{<\min\{2k-k_2,j-C\}}\phi_2)||_{\dot{X}_k^{-\epsilon,\epsilon,\infty}}\\&\leq C2^{\epsilon(j-k)}|j-k|^2||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.\end{aligned}$$

The remaining terms of (11) are estimated by means of (8): for example

$$\begin{aligned}&||P_kQ_j(P_{k_1}Q_{j-C\leq .<2k-k_1}\phi_1P_{k_2}Q_{<2k-k_2}\phi_2)||_{L_t^2L_x^2}\\&\leq C_\mu 2^{\frac{3k}{2}}2^{\frac{j-k}{2+\mu}}||P_{k_1}Q_{j-C\leq .<2k-k_1}\phi_1||_{L_t^2L_x^2}||P_{k_2}\phi_2||_{L_t^\infty L_x^2}\\&\leq C_\mu 2^{\frac{3k-j-2k_1}{2}}2^{\frac{j-k}{2+\mu}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]},\end{aligned}$$

where $\mu > 0$ is arbitrary. This entails the estimate

$$||P_kQ_j(P_{k_1}Q_{j-C\leq .<2k-k_1}\phi_1 P_{k_2}Q_{<2k-k_2}\phi_2)||_{\dot{X}_k^{-\epsilon,\epsilon,0}} \leq C_{\epsilon,\delta(\epsilon)}2^{k-k_1}2^{\delta(\epsilon)(j-k)}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]},$$

where $0 < \delta(\epsilon) < \epsilon$. We are done with the high-high case.

(2) *High-Low interactions:* $k_1 + 5 \geq k \geq k_1 - 5$.

As before we decompose

$$P_k(P_{k_1}\phi_1 P_{k_2}\phi_2) = P_kQ_{\geq k+10}(P_{k_1}\phi_1 P_{k_2}\phi_2) + P_kQ_{<k+10}(P_{k_1}\phi_1 P_{k_2}\phi_2). \tag{12}$$

We begin with the first term on the right-hand side. Freeze its modulation to size $\sim 2^j$, $j \geq k+10$. Then observe that

$$P_kQ_j(P_{k_1}\phi_1 P_{k_2}\phi_2) = P_kQ_j(P_{k_1}Q_{\geq j-C}\phi_1 P_{k_2}\phi_2) + P_kQ_j(P_{k_1}Q_{<j-C}\phi_1 P_{k_2}Q_{\geq j-C}\phi_2). \tag{13}$$

We have

$$\begin{aligned}&||P_kQ_j(P_{k_1}Q_{\geq j-C}\phi_1 P_{k_2}\phi_2)||_{L_t^2L_x^2} \leq C||P_{k_1}Q_{\geq j-C}\phi_1||_{L_t^2L_x^2}||P_{k_2}\phi_2||_{L_t^\infty L_x^\infty}\\ &\leq C2^{k_2-\frac{k_1+j}{2}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.\end{aligned}$$

The 2nd term on the right-hand side of (13) is estimated similarly and left out. We proceed to the 2nd term of (12). Freeze its modulation to dyadic value $\sim 2^j$, $j < k+10$. Then use a decomposition

$$\begin{aligned}&P_kQ_j(P_{k_1}\phi_1 P_{k_2}\phi_2) = P_kQ_j(P_{k_1}Q_{\geq j-C}\phi_1 P_{k_2}Q_{<j-C}\phi_2)\\ &+ P_kQ_j(P_{k_1}\phi_1 P_{k_2}Q_{\geq j-C}\phi_2) + P_kQ_j(P_{k_1}Q_{<j-C}\phi_1 P_{k_2}Q_{<j-C}\phi_2).\end{aligned} \tag{14}$$

We commence by estimating the first term on the right-hand side. We write

$$P_kQ_j(P_{k_1}Q_{\geq j-C}\phi_1 P_{k_2}Q_{<j-C}\phi_2) = \sum_{l\geq j-C} P_kQ_j(P_{k_1}Q_l\phi_1 P_{k_2}Q_{<j-C}\phi_2).$$

First assume $j+10 > l \geq j-C$. We distinguish between the case $j \geq k_2 - 10$ and the opposite. In the first situation, we have

$$||P_kQ_j(P_{k_1}Q_l\phi_1 P_{k_2}Q_{<j-C}\phi_2)||_{L_t^2L_x^2} \leq C2^{k_2-\frac{j+k_1}{2}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.$$

The desired inequality follows easily form this. In the case $j < k_2 - 10$, elementary geometry considerations imply that

$$\begin{aligned}&P_kQ_j(P_{k_1}Q_l\phi_1P_{k_2}Q_{<j-C}\phi_2)\\&=\sum_{\pm,\pm}\sum_{\kappa_{1,2}\in K_{\frac{j-k_2}{2}-10},\,\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\leq 2^{\frac{j-k_2}{2}+O(1)}}P_kQ_j(P_{k_1,\kappa_1}Q_l^{\pm}\phi_1P_{k_2,\kappa_2}Q^{\pm}_{<j-C}\phi_2).\end{aligned}$$

Therefore, using the classical Bernstein's inequality (7), Cauchy-Schwarz, Plancherel's theorem and the definition of the $S[k]$, we obtain

$$\begin{aligned}&||P_kQ_j(P_{k_1}Q_l\phi_1P_{k_2}Q_{<j-C}\phi_2)||_{L_t^2L_x^2}\\&\leq C2^{\frac{3k_2}{2}}2^{\frac{j-k_2}{2}}\sum_{\pm,\pm}\sum_{\kappa_{1,2}\in K_{\frac{j-k_2}{2}-10},\,\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\leq 2^{\frac{j-k_2}{2}+O(1)}}\\&\qquad ||P_{k_1,\kappa_1}Q_l^{\pm}\phi_1||_{L_t^2L_x^2}||P_{k_2,\kappa_2}Q^{\pm}_{<j-C}\phi_2||_{L_t^\infty L_x^2}\\&\leq C2^{\frac{k_2-k_1}{2}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.\end{aligned}$$

The desired inequality follows from this.
Next, assume $l\geq j+10$. Elementary geometric considerations imply $l<k_2+O(1)$, as well as the decomposition

$$\begin{aligned}&P_kQ_j(P_{k_1}Q_l\phi_1P_{k_2}Q_{<j-C}\phi_2)\\&=\sum_{\pm,\pm}\sum_{\kappa_{1,2}\in K_{\frac{l-k_2}{2}-C},\,\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{l-k_2}{2}}}P_kQ_j(P_{k_1,\kappa_1}Q_l^{\pm}\phi_1P_{k_2}Q^{\pm}_{<j-C}\phi_2).\end{aligned}$$

Now one argues exactly as in the immediately preceding, obtaining the bound

$$\begin{aligned}&||P_kQ_j(P_{k_1}Q_l\phi_1P_{k_2}Q_{<j-C}\phi_2)||_{\dot{X}_k^{-\epsilon,\epsilon,\infty}}\\&\qquad\leq C2^{\epsilon(j-k_1)}2^{\frac{l-k_2}{2}}2^{-\frac{k_1+l}{2}}2^{k_2}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.\end{aligned}$$

This can be summed over $j+O(1)<l<k_2+O(1)$ to yield the desired inequality.

We proceed to the 2nd summand on the right-hand side of (14). This is estimated by means of the improved Bernstein's inequality:

$$\begin{aligned}&||P_kQ_j(P_{k_1}\phi_1P_{k_2}Q_{\geq j-C}\phi_2)||_{L_t^2L_x^2}\leq C||P_{k_1}\phi_1||_{L_t^\infty L_x^2}||P_{k_2}Q_{\geq j-C}\phi_2||_{L_t^2L_x^\infty}\\&\leq C2^{-\frac{j}{2}}2^{k_2-\frac{k_1}{2}}2^{\min\{\frac{j-k_2}{2+},0\}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.\end{aligned}$$

The desired inequality follows immediately from this.

As for the last term of (14), we proceed as in the high-high case:

$$
\begin{aligned}
&P_kQ_j(P_{k_1}Q_{<j-C}\phi_1 P_{k_2}Q_{<j-C}\phi_2)\\
&=\sum_{\pm,\pm}\sum_{\kappa_{1,2}\in K_{\frac{j-k_2}{2}-10},\,\mathrm{dist}(\pm\kappa_1,\pm\kappa_2)\sim 2^{\frac{j-k_2}{2}}} P_kQ_j(P_{k_1,\kappa_1}Q^{\pm}_{<j-C}\phi_1 P_{k_2,\kappa_2}Q^{\pm}_{<j-C}\phi_2).
\end{aligned}
$$

Now we use (5), as well as Cauchy-Schwarz and the fact that

$$
\Big(\sum_{\kappa\in K_{\frac{j-k_2}{2}-10}}||P_{k_1,\kappa}Q^{\pm}_{<j-C}\phi_1||^2_{S[k_1,\pm\kappa]}\Big)^{\frac{1}{2}}\leq\Big(\sum_{\kappa\in K_{\frac{j-k_1}{2}-10}}||P_{k_1,\kappa}Q^{\pm}_{<j-C}\phi_1||^2_{S[k_1,\pm\kappa]}\Big)^{\frac{1}{2}},
$$

which is a consequence of the *fundamental orthogonality relation* satisfied by the $S[k,\kappa]$: provided $\phi_\kappa\in\mathcal{S}(\mathbf{R}^{3+1})$ has Fourier support contained in an angular sector $\kappa\subset S^2$, $\kappa\in K_l$ and at frequency 2^k and provided the modulation of ϕ is of size $<2^{k+2l'}$, $l'<<l$, then the following holds:

$$
\phi_\kappa=\sum_{\kappa'\subset\kappa,\,\kappa'\in K_{l'}}\phi_{\kappa'}\rightarrow||\phi_\kappa||_{S[k,\kappa]}\leq C\Big(\sum_{\kappa'\in K_{l'}}||\phi_{\kappa'}||^2_{S[k,\kappa']}\Big)^{\frac{1}{2}},
$$

where the $\phi_{\kappa'}$ have Fourier support contained in the angular sector $\kappa'\subset S^2$, see [7], [12]. We conclude that

$$
||P_kQ_j(P_{k_1}Q_{<j-C}\phi_1P_{k_2}Q_{<j-C}\phi_2)||_{L^2_tL^2_x}\leq C2^{\frac{k_2-k_1}{2}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.
$$

From this the desired inequality follows easily. We are done with high-low interactions. The case of *low-high interactions* is of course exactly the same. ■

The preceding theorem may seem superfluous, as there is an $L^4_tL^4_x$-Strichartz estimate in $n=3$ spatial dimensions. However, the fact that we substitute null-frame spaces ($NFA[\kappa]$) averaged over cap decompositions for the customary energy space $L^1_tL^2_x$ appears to either render this norm unavailable[10] or rather difficult to prove. The author has been unable thus far to build this norm into $S[k]$.

Note that the reason why we cannot place the product of two functions into $L^2_tL^2_x$ has to do with the fact that we cannot gain exponentially in the angle between the Fourier supports. However, this gain is obviously present in case of the null-form $Q_{\nu j}(\phi_1,\phi_2)=R_\nu\phi_1R_j\phi_2-R_j\phi_1R_\nu\phi_2$. Thus the following theorem is not at all surprising:

Theorem 3.2. *Let ϕ_1,ϕ_2 be Schwartz functions. Then the following inequality holds:*

[10] in order to keep the energy inequality valid.

$$||P_k(R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)||_{L_t^2 L_x^2} \\ \leq C 2^{-\frac{|k_1-k_2|}{2}} 2^{k-\max\{k_1,k_2\}} ||P_{k_1}\phi_1||_{S[k_1]} ||P_{k_2}\phi_2||_{S[k_2]}.$$

Remark: The proof will actually reveal an inequality of the form

$$||P_k(D_+ D_-^{-1})^\lambda (R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)||_{L_t^2 L_x^2} \\ \leq C_\lambda 2^{(\lambda-\frac{1}{2})|k_1-k_2|} 2^{k-\max\{k_1,k_2\}} ||P_{k_1}\phi_1||_{S[k_1]} ||P_{k_2}\phi_2||_{S[k_2]},$$

where $0 \leq \lambda < \frac{1}{4}$ and D_+, D_-, are given by the homogeneous symbols $|\tau| + |\xi|$, $|\,|\tau| - |\xi|\,|$, respectively.

Proof :

(1): *High-Low interactions.*: $k_1 = k + O(1) \geq k_2 + O(1)$.

We first assume $\nu = i$. We decompose the function $Q_{ij}(P_{k_1}\phi_1, P_{k_2}\phi_2)$ into a sum of terms: Let $\chi_r(.)$, $r \in \mathbf{Z}$ be a smooth bump function localizing to an interval of length $\sim 2^r$ around 2^r, such that

$$\sum_{r \in \mathbf{Z}} \chi_r(x) = 1,\ x \in (0, \infty).$$

Now we write

$$\mathcal{F}P_k(R_i P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_i P_{k_2}\phi_2)(\zeta) \\ = \sum_{r<O(1)} m_k(\zeta) \int_{\eta+\xi=\zeta} \chi_r(<\xi,\eta>) m_{k_1}(\xi) m_{k_2}(\eta) (\frac{\xi_i}{|\xi|}\frac{\eta_j}{|\eta|} - \frac{\xi_j}{|\xi|}\frac{\eta_i}{|\eta|}) \tilde{\phi}_1(\xi)\tilde{\phi}_2(\eta) d\eta.$$

In this context, η, ξ, refer to space-time coordinates in Fourier space, while $<\xi, \eta>$ refers to the angle between $\frac{\xi_0}{|\xi_0|}\xi$, $\frac{\eta_0}{|\eta_0|}\eta$ (omit integration over the planes $\xi_0 = 0$, $\eta_0 = 0$). W. l. o. g. we restrict attention to the case $\xi_0 > 0$, $\eta_0 > 0$, i. e. $\phi_{1,2}$ microlocalized to the upper half-space $\tau > 0$. Observe that we have

$$\chi_r(<\xi,\eta>) m_{k_1}(\xi) m_{k_2}(\eta) (\frac{\xi_i}{|\xi|}\frac{\eta_j}{|\eta|} - \frac{\xi_j}{|\xi|}\frac{\eta_i}{|\eta|}) \tilde{\phi}_1(\xi)\tilde{\phi}_2(\eta) \\ = \sum_{\kappa_{1,2} \in K_{r-10},\, \mathrm{dist}(\kappa_1,\kappa_2) \sim 2^r} \chi_r(<\xi,\eta>) m_{k_1}(\xi) m_{k_2}(\eta) \\ a_{\kappa_1}(\frac{\xi}{|\xi|}) a_{\kappa_2}(\frac{\eta}{|\eta|}) (\frac{\xi_i}{|\xi|}\frac{\eta_j}{|\eta|} - \frac{\xi_j}{|\xi|}\frac{\eta_i}{|\eta|}) \tilde{\phi}_1(\xi)\tilde{\phi}_2(\eta),$$

where $a_{\kappa_{1,2}}(.)$ were introduced in section 2. Now observe that the function

$$(\xi, \eta) \to \Lambda_{r,\kappa_1,\kappa_2}(\xi, \eta) := \chi_r(<\xi,\eta>) m_{k_1}(\xi) m_{k_2}(\eta) a_{\kappa_1}(\frac{\xi}{|\xi|}) a_{\kappa_2}(\frac{\eta}{|\eta|}) (\frac{\xi_i}{|\xi|}\frac{\eta_j}{|\eta|} - \frac{\xi_j}{|\xi|}\frac{\eta_i}{|\eta|})$$

is smooth and compactly supported; hence we can expand it into a (discrete) Fourier series. We need to estimate the l^1-norm of its coefficients. We change coordinates $(\xi,\eta)\to(\xi',\eta')$ in such a way that one coordinate axis is perpendicular to κ_1 while the others are 'tangential'. Then we expand

$$\Lambda_{r,\kappa_1,\kappa_2}(\xi,\eta)=\sum_{n\in 2^{-k_1}\mathbf{Z}^3,\, m\in 2^{-k_2}\mathbf{Z}^3} a_{n,m,\kappa_1,\kappa_2}e^{2\pi(n\cdot\xi'+m\cdot\eta')}.$$

There are two ways to estimate the $a_{n,m,\kappa_1,\kappa_2}$: the obvious estimate is obtained by using the L^∞-norm of $\Lambda_{r,\kappa_1,\kappa_2}$. This furnishes the estimate

$$|a_{n,m,\kappa_1,\kappa_2}|\leq C\mathrm{dist}(\kappa_1,\kappa_2)|\kappa_1||\kappa_2|\leq C2^{5r}.$$

Alternatively, we can use integration by parts. Note that there are two 'bad directions', namely the coordinate directions tangential to the cap κ_1. Upon writing $n=(n_1,n_2,n_3)$ etc., we obtain

$$\begin{aligned}&|a_{n,m,\kappa_1,\kappa_2}|\\ &\leq C2^{-4r}\prod_{i=1,2}(1+2^{k_1}|n_i|)^{-2}\prod_{i=1,2}(1+2^{k_2}|m_i|)^{-2}(1+2^{k_1}|n_3|)^{-M}(1+2^{k_2}|m_3|)^{-M}.\end{aligned}$$

From the preceding two estimates we conclude that

$$\begin{aligned}|a_{n,m,\kappa_1,\kappa_2}|\leq C2^{5r\frac{1-\delta}{2}}2^{-4r\frac{1+\delta}{2}}\prod_{i=1,2}(1+2^{k_1}|n_i|)^{-(1+\delta)}\prod_{i=1,2}(1+2^{k_2}|m_i|)^{-(1+\delta)}\\ (1+2^{k_1}|n_3|)^{-M}(1+2^{k_2}|m_3|)^{-M}.\end{aligned}$$

Represent the linear coordinate transformation $\xi\to\xi'$ by the matrix A_{κ_1,κ_2}. Then we conclude that

$$\begin{aligned}&P_k(R_iP_{k_1}\phi_1R_jP_{k_2}\phi_2-R_jP_{k_1}\phi_1R_iP_{k_2}\phi_2)\\ &=\sum_{r<O(1)}\sum_{\kappa_{1,2}\in K_{r-10},\,\mathrm{dist}(\kappa_1,\kappa_2)\sim 2^r}\sum_{n\in 2^{-k_1}\mathbf{Z}^3,\, m\in 2^{-k_2}\mathbf{Z}^3}\\ &\qquad a_{n,m,\kappa_1,\kappa_2}(P_{k_1,\kappa_1}\phi_1)(.-A_{\kappa_1,\kappa_2}n)(P_{k_2,\kappa_2}\phi_2)(.-A_{\kappa_1,\kappa_2}m).\end{aligned}$$

This is estimated as in the proof of the preceding theorem. One decomposes $(P_{k_1,\kappa_1}\phi_1)(.-A_{\kappa_1,\kappa_2}n)(P_{k_2,\kappa_2}\phi_2)(.-A_{\kappa_1,\kappa_2}m)$ into terms of the form

$$(P_{k_1,\kappa_1}Q_{<\geq k_1+2r}\phi_1)(.-A_{\kappa_1,\kappa_2}n)(P_{k_2,\kappa_2}Q_{<\geq k_2+2r}\phi_2)(.-A_{\kappa_1,\kappa_2}m).$$

Each of these can be estimated in $L_t^2L_x^2$ as in the preceding proof[11]; the only slightly

[11] One also invokes the translation invariance of all Banach spaces used.

different term is

$$(P_{k_1,\kappa_1}Q_{\geq k_1+2r}\phi_1)(.-A_{\kappa_1,\kappa_2}n)(P_{k_2,\kappa_2}Q_{\geq k_2+2r}\phi_2)(.-A_{\kappa_1,\kappa_2}m).$$

It is estimated as follows:

$$\begin{aligned}&||(P_{k_1,\kappa_1}Q_{\geq k_1+2r}\phi_1)(.-A_{\kappa_1,\kappa_2}n)(P_{k_2,\kappa_2}Q_{\geq k_2+2r}\phi_2)(.-A_{\kappa_1,\kappa_2}m)||_{L_t^2L_x^2}\\ &\leq \sum_{a_i\geq k_i+2r,\, i=1,2}||(P_{k_1,\kappa_1}Q_{a_1}\phi_1)(.-A_{\kappa_1,\kappa_2}n)(P_{k_2,\kappa_2}Q_{a_2}\phi_2)(.-A_{\kappa_1,\kappa_2}m)||_{L_t^2L_x^2}\\ &\leq C2^{\frac{k_2-k_1}{2}}\sum_{a_i\geq k_i+2r,\, i=1,2}2^{\frac{r}{2}+\frac{k_1-a_1}{4}}2^{\frac{r}{2}+\frac{k_2-a_2}{4}}\prod_{i=1,2}||P_{k_i,\kappa_i}Q_{a_i}\phi_i||_{\dot{X}_{k_i}^{\frac{1}{2},\frac{1}{2},\infty}}.\end{aligned}$$

The conclusion is that

$$||(P_{k_1,\kappa_1}\phi_1)(.-A_{\kappa_1,\kappa_2}n)(P_{k_2,\kappa_2}\phi_2)(.-A_{\kappa_1,\kappa_2}m)||_{L_t^2L_x^2}\leq C2^{\frac{k_2-k_1}{2}}\prod_{i=1,2}M_{\phi_i,\kappa_i},$$

where we let

$$M_{\phi_i,\kappa_i}=||P_{k_i,\kappa_i}Q_{<k_i+2r}\phi_i||_{S[k_i,\kappa_i]}+\sum_{a\geq k_i+2r}2^{\frac{k_i+2r-a}{4}}||P_{k_i,\kappa_i}Q_a\phi_i||_{\dot{X}_{k_i}^{-\frac{1}{2},\frac{1}{2},\infty}}.$$

Finally

$$\begin{aligned}&||P_k(R_iP_{k_1}\phi_1R_jP_{k_2}\phi_2-R_jP_{k_1}\phi_1R_iP_{k_2}\phi_2)||_{L_t^2L_x^2}\\ &\leq C2^{\frac{k_2-k_1}{2}}\sum_{r<O(1)}\sum_{\kappa_{1,2}\in K_{r-10},\,\mathrm{dist}(\kappa_1,\kappa_2)\sim 2^r}\sum_{n\in 2^{-k_1}\mathbf{Z}^3,\, m\in 2^{-k_2}\mathbf{Z}^3}a_{n,m,\kappa_1,\kappa_2}\prod_{i=1,2}M_{\phi_i,\kappa_i}\\ &\leq C2^{\frac{k_2-k_1}{2}}\sum_{r<O(1)}2^{r\frac{1-9\delta}{2}}\prod_{i=1,2}(\sum_{\kappa_i\in K_{r-10}}M_{\phi_i,\kappa_i}^2)^{\frac{1}{2}}\leq C2^{\frac{k_2-k_1}{2}}\prod_{i=1,2}||P_{k_i}\phi_i||_{S[k_i]}.\end{aligned}$$

We still have to settle the case $\nu=0$. We shall again assume w. l. o. g. that ϕ_1, ϕ_2 are microlocalized to the upper half-space $\tau>0$. We write

$$R_0\phi_1R_j\phi_2-R_j\phi_1R_0\phi_2=[(R_0-1)\phi_1R_j\phi_2-R_j\phi_1(R_0-1)\phi_2]+[\phi_1R_j\phi_2-R_j\phi_1\phi_2].$$

The 2nd $[,]$ is dealt with precisely as in the preceding case. As to the first, we observe that the operator R_0-1 has symbol $\frac{\tau}{|\xi|}-1$, whence

$$||(R_0-1)P_{k_1}\phi_1||_{L_t^2L_x^2}\leq C2^{-k_1}||P_{k_1}\nabla_{x,t}\phi_1||_{\dot{X}_{k_1}^{-\frac{1}{2},\frac{1}{2},\infty}}.$$

The desired estimate follows easily from this.

(2): *High-High interactions*: $k_1 = k_2 + O(1) > k + O(1)$.

They are handled similarly, see also the proof of the preceding theorem. We leave the details for the reader. ∎

The next theorem deals with the (stronger) $Q_0(u,v) = \partial_\nu u \partial^\nu v$ null-form. One uses the elementary identity

$$2\partial_\nu u \partial^\nu v = \Box(uv) - \Box u v - u \Box v :$$

Theorem 3.3. *Let ϕ, ψ be Schwartz functions. Then the following inequalities hold:*

$$||P_k \nabla_x [R_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\psi_2]||_{N[k]} \leq C||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]},$$

$$||P_k [R_\nu P_{k_1}\phi_1 \partial^\nu P_{k_2}\phi_2]||_{N[k]} \leq C2^{\delta \min\{k-\max\{k_1,k_2\},0\}}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]},$$

for appropriate $\delta > 0$. Finally, we have the inequality

$$||R_\nu \phi R^\nu \psi||_{L_t^2 L_x^2} \leq C(\sum_{k\in\mathbf{Z}} ||P_k\phi||_{S[k]}^2)^{\frac{1}{2}}(\sum_{k\in\mathbf{Z}} ||P_k\phi||_{S[k]}^2)^{\frac{1}{2}}.$$

Remark: The first inequality is contained in [12]. The 2nd is stronger than the first for high-high interactions.

Proof : *The 2nd inequality:* We only prove the 2nd inequality for high-high interactions, i. e. $k_1 = k_2 + O(1) \geq k + O(1)$. We may rescale $k = 0$, and also assume that $k_1 \geq 100$. First, it is easy to see that both inputs may be reduced to modulation $< 2^{k_1-10}$: For example we have

$$\begin{aligned}
&||P_0 Q_{\geq k_1-15}[R_\nu P_{k_1} Q_{\geq k_1-10}\phi_1 \partial^\nu P_{k_2}\phi_2]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}} \\
&\leq C2^{-\frac{k_1}{2}}||R_\nu P_{k_1} Q_{\geq k_1-10}\phi_1||_{L_t^2 L_x^2}||\partial_\nu P_{k_2}\phi_2||_{L_t^\infty L_x^2} \\
&\leq C2^{-k_1}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&||P_0 Q_{<k_1-15}[R_\nu P_{k_1} Q_{\geq k_1-10}\phi_1 \partial^\nu P_{k_2}\phi_2]||_{L_t^1 L_x^2} \\
&= ||P_0 Q_{<k_1-15}[R_\nu P_{k_1} Q_{\geq k_1-10}\phi_1 \partial^\nu P_{k_2} Q_{\geq k_1-15}\phi_2]||_{L_t^1 L_x^2} \\
&\leq C||R_\nu P_{k_1} Q_{\geq k_1-10}\phi_1||_{L_t^2 L_x^2}||\partial^\nu P_{k_2} Q_{\geq k_1-15}\phi_2||_{L_t^2 L_x^2} \leq C2^{-k_1}\prod_{i=1,2}||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}$$

Next we distinguish between the following cases:

(1): *Inputs microlocalized to the same half-space* $\tau >< 0$. Assume w. l. o. g. that this is the half-space $\tau > 0$. Then the modulation of the output is $\sim 2^{k_1}$. We have the decomposition

$$
\begin{aligned}
&P_0[P_{k_1}Q^+_{<k_1-10}R_\nu\phi_1 P_{k_2}Q^+_{<k_1-10}\partial^\nu\phi_2]\\
&=\sum_{\kappa_{1,2}\in K_{-k_1-10},\,\mathrm{dist}(\kappa_1,-\kappa_2)\leq 2^{-k_1+O(1)}} P_0Q_{k_1+O(1)}[P_{k_1,\kappa_1}Q^+_{<k_1-10}R_\nu\phi_1\\
&\qquad\qquad P_{k_2,\kappa_2}Q^+_{<k_1-10}\partial^\nu\phi_2].
\end{aligned}
$$

Now we can use (5), as well as the definition of the $S[k]$:

$$
\begin{aligned}
&||P_0[P_{k_1}Q^+_{<k_1-10}R_\nu\phi_1 P_{k_2}Q^+_{<k_1-10}\partial^\nu\phi_2]||_{N[0]}\\
&\sum_{\kappa_{1,2}\in K_{-k_1-10},\,\mathrm{dist}(\kappa_1,-\kappa_2)\leq 2^{-k_1+O(1)}} ||P_0Q_{k_1+O(1)}[P_{k_1,\kappa_1}Q^+_{<k_1-10}R_\nu\phi_1\\
&\qquad\qquad P_{k_2,\kappa_2}Q^+_{<k_1-10}\partial^\nu\phi_2]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}}\\
&\leq C2^{-\frac{k_1}{2+}}\prod_{i=1,2}||P_{k_i}\phi_1||_{S[k_i]}.
\end{aligned}
$$

(2): *Both inputs microlocalized to different half-spaces.* We may assume that the modulation of the output is $< 2^C$. For assume the opposite, i. e. the output is at modulation $\sim 2^j$, $j \geq C$. Then at least one of the inputs is at modulation $> 2^{j-10}$. For example, assume the first input satisfies this condition. Use the identity

$$
\begin{aligned}
&P_0Q_j[R_\nu P_{k_1}Q^+_{j-10\leq .<k_1-10}\phi_1 P_{k_2}Q^-_{<k_1-10}\partial^\nu\phi_2]\\
&= P_0Q_j\partial^\nu[R_\nu P_{k_1}Q^+_{j-10\leq .<k_1-10}\phi_1 P_{k_2}Q^-_{<k_1-10}\phi_2]\\
&- P_0Q_j[\Box\nabla^{-1}P_{k_1}Q^+_{j-10\leq .<k_1-10}\phi_1 P_{k_2}Q^-_{<k_1-10}\phi_2].
\end{aligned}
$$

We estimate each of the terms on the right-hand side:

$$
\begin{aligned}
&||P_0Q_j\partial^\nu[R_\nu P_{k_1}Q^+_{j-10\leq .<k_1-10}\phi_1 P_{k_2}Q^-_{<k_1-10}\phi_2]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}}\\
&\leq C2^{\frac{j}{2}}||R_\nu P_{k_1}Q^+_{j-10\leq .<k_1-10}\phi_1||_{L_t^2L_x^2}||P_{k_2}Q^-_{<k_1-10}\phi_2||_{L_t^\infty L_x^2}\\
&\leq C2^{-k_1}\prod_{i=1,2}||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

This is more than enough. Similarly, we have

$$\begin{aligned}&||P_0Q_j[\Box\nabla^{-1}P_{k_1}Q^{+}_{j-10\leq .<k_1-10}\phi_1 P_{k_2}Q^{-}_{<k_1-10}\phi_2]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}}\\&\leq C2^{-\frac{j}{2}}||\Box\nabla^{-1}P_{k_1}Q^{+}_{j-10\leq .<k_1-10}\phi_1||_{L^2_tL^2_x}||P_{k_2}Q^{-}_{<k_1-10}\phi_2||_{L^\infty_tL^2_x}\\&\leq C2^{-\frac{j}{2}}2^{-\frac{k_1}{2}}\prod_{i=1,2}||P_{k_i}\phi_i||_{S[k_i]}.\end{aligned}$$

Having reduced the modulation of the output and the inputs, we now proceed to expand the null-form. We need to estimate the following 3 terms:

(A): $\Box P_0Q_{<C}[\nabla^{-1}P_{k_1}Q^{+}_{<k_1-10}\phi_1 P_{k_2}Q_{<k_1-10}\phi_2]$. Use theorem 3.1:

$$\begin{aligned}&||\Box P_0Q_{<C}[\nabla^{-1}P_{k_1}Q^{+}_{<k_1-10}\phi_1 P_{k_2}Q^{-}_{<k_1-10}\phi_2]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}}\\&\leq C||P_0Q_{<C}[\nabla^{-1}P_{k_1}Q^{+}_{<k_1-10}\phi_1 P_{k_2}Q^{-}_{<k_1-10}\phi_2]||_{\dot{X}_0^{\frac{1}{2},\frac{1}{2},1}}\\&\leq C2^{-k_1}\prod_{i=1,2}||P_{k_i}\phi_i||_{S[k_i]}.\end{aligned}$$

(B): $P_0Q_{<C}[\nabla^{-1}P_{k_1}Q_{<k_1-10}\Box\phi_1 P_{k_2}Q_{<k_1-10}\phi_2]$. Use lemma 2.1: We compute

$$\begin{aligned}&||P_0Q_{<C}[\nabla^{-1}P_{k_1}Q_{<k_1-10}\Box\phi_1 P_{k_2}Q_{<k_1-10}\phi_2]||_{N[0]}\\&\leq C2^{-\delta_1k_1}\sum_{j<k_1-10}2^{\delta_2(j-k_1)}||\nabla^{-1}P_{k_1}Q_j\Box\phi_1||_{\dot{X}_{k_1}^{\frac{1}{2},-\frac{1}{2},\infty}}||P_{k_2}Q_{<k_1-10}\phi_2||_{S[k_2]}\\&\leq C2^{-\delta_1k_1}\prod_{i=1,2}||P_{k_i}\phi_i||_{S[k_i]}.\end{aligned}$$

(C): $P_0Q_{<C}[\nabla^{-1}P_{k_1}Q_{<k_1-10}\phi_1\Box P_{k_2}Q_{<k_1-10}\phi_2]$. This is estimated exactly like the preceding term.

The third inequality: It follows from the more concise inequality

$$||P_{k_1}R_\nu\phi_1P_{k_2}R^\nu\phi_2||_{L^2_tL^2_x}\leq C2^{-\delta|k_1-k_2|}||P_{k_1}\phi_1||_{S[k_1]}||P_{k_2}\phi_2||_{S[k_2]}$$

for suitable $\delta>0$. This inequality is proved similarly to the preceding one and left for the reader. ■

4. Trilinear estimates

We commence with the following fairly simple estimates. They are (essentially) the analogue for $n = 3$ of a more difficult estimate in [12]. The proof here is of course much simpler:

Theorem 4.1. *Let ϕ_i, $i = 1, 2, 3,$ be Schwartz functions. Then we have the inequalities*

$$||P_0[R_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\phi_2 P_{k_3}\phi_3]||_{N[0]}$$
$$\leq C2^{-\delta_1|k_1-k_2|}2^{\delta_2 \min\{k_3-\max\{k_1,k_2\},0\}}2^{-\delta_3|k_3|}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]},$$

$$||P_0[\nabla^{-1}(\partial_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\phi_2)P_{k_3}\phi_3]||_{N[0]}$$
$$\leq C2^{-\delta_1|k_1-k_2|}2^{\delta_2 \min\{k_3-\max\{k_1,k_2\},0\}}2^{-\delta_3|k_3|}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]},$$

for appropriate $\delta_i > 0$, $i = 1, 2, 3$.

Proof We prove the 2nd inequality, as the first is proved similarly.

(A): *High-High interactions:* $P_0[\nabla^{-1}P_{k_3+O(1)}(\partial_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\phi_2)P_{k_3}\phi_3]$, $k_3 > 10$. We further distinguish between the following cases:

(A.1): $k_1 = k_2 + O(1)$. Use theorem 3.3 as well as lemma 2.1:

$$||P_0[\nabla^{-1}P_{k_3+O(1)}(\partial_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\phi_2)P_{k_3}\phi_3]||_{N[0]}$$
$$\leq C2^{-\delta_1 k_3}||P_{k_3+O(1)}(\partial_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\phi_2)||_{N[k_3+O(1)]}||P_{k_3}\phi_3||_{S[k_3]}$$
$$\leq C2^{-\delta_1 k_3}2^{\delta_2(k_3-k_1)}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.$$

(A.2): $k_1 = k_3 + O(1)$, $-k_1 < k_2 \leq k_1 + O(1)$. This is handled like the preceding case. One incurs an inessential logarithmic loss $C|k_1|$.

(A.3): $k_2 \leq -k_1$. We need to distinguish between different cases involving modulations:
(A.3.1): $P_{k_1}\partial_\nu\phi_1$ *at modulation* $\geq 2^{k_2}$*:* note that provided $p > 4$, using Bernstein's inequality (7)

$$\begin{aligned}&||P_0[\nabla^{-1}P_{k_3+O(1)}(P_{k_1}Q_{\geq k_2}\partial_\nu\phi_1 R^\nu P_{k_2}\phi_2)P_{k_3}\phi_3]||_{L_t^1L_x^2}\\&\leq C2^{-k_1}||P_{k_1}Q_{\geq k_2}\partial_\nu\phi_1||_{L_t^2L_x^2}||R^\nu P_{k_2}\phi_2||_{L_t^4L_x^\infty}||P_{k_3}\phi_3||_{L_t^4L_x^p}\\&\leq C_p 2^{\frac{k_2}{4}-\frac{k_1}{2}}2^{k_1\frac{3p-12}{4p}}\prod_i||P_{k_i}\phi_i||_{S[k_i]}.\end{aligned}$$

This is acceptable provided $p>4$ is small enough.

(A.3.2): *$P_{k_2}\phi_2$ at modulation $\geq 2^{k_2+\mu k_1}$, $\mu>0$, $P_{k_1}\phi_1$ at modulation $<2^{k_2}$.* This is similar to the preceding term:

$$\begin{aligned}&||P_0[\nabla^{-1}P_{k_3+O(1)}(P_{k_1}Q_{<k_2}\partial_\nu\phi_1 R^\nu P_{k_2}Q_{\geq k_2+\mu k_1}\phi_2)P_{k_3}\phi_3]||_{L_t^1L_x^2}\\&\leq C||P_{k_1}Q_{<k_2}\partial_\nu\phi_1||_{L_t^4L_x^p}||P_{k_3}\phi_3||_{L_t^4L_x^p}||R^\nu P_{k_2}Q_{\geq k_2+\mu k_1}\phi_2||_{L_t^2L_x^M}\\&\leq C2^{\frac{k_2}{2+}}2^{-\frac{k_1\mu}{2+}}\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]},\end{aligned}$$

where we let $\frac{2}{p}+\frac{1}{M}=\frac{1}{2}$, and $p>4$ is sufficiently close to 4.

(A.3.3): Having reduced the modulations, we can now expand the null-structure. We need to estimate the following terms:

(I): $P_0[\nabla^{-1}\Box(\nabla^{-1}P_{k_2}Q_{<k_2+\mu k_1}\phi_2 P_{k_1}Q_{<k_2}\phi_1)P_{k_3}\phi_3]$. Observe that $(,)$ is at modulation $2^{k_2+\mu k_1+O(1)}$. Now use lemma 2.1, theorem 3.2:

$$\begin{aligned}&||P_0[\nabla^{-1}\Box(\nabla^{-1}P_{k_2}Q_{<k_2+\mu k_1}\phi_2 P_{k_1}Q_{<k_2}\phi_1)P_{k_3}\phi_3]||_{N[0]}\\&\leq C\sum_{j<k_2+\mu k_1+O(1)}||P_0[\nabla^{-1}\Box Q_j(\nabla^{-1}P_{k_2}Q_{<k_2+\mu k_1}\phi_2 P_{k_1}Q_{<k_2}\phi_1)P_{k_3}\phi_3]||_{N[0]}\\&\leq C\sum_{j<k_2+\mu k_1+O(1)}2^{\delta(j-k_1)}\prod_i||P_{k_i}\phi_i||_{S[k_i]}.\end{aligned}$$

This is acceptable provided $\mu<1$.

(II): $P_0[\nabla^{-1}(\nabla^{-1}P_{k_2}Q_{<k_2+\mu k_1}\phi_2 P_{k_1}Q_{<k_2}\Box\phi_1)P_{k_3}\phi_3]$. As before let $p>4$. Then we estimate

$$\begin{aligned}&||P_0[\nabla^{-1}(\nabla^{-1}P_{k_2}Q_{<k_2+\mu k_1}\phi_2 P_{k_1}Q_{<k_2}\Box\phi_1)P_{k_3}\phi_3]||_{L_t^1L_x^2}\\&\leq C2^{-k_1}||P_{k_1}Q_{<k_2}\Box\phi_1||_{L_t^2L_x^2}||\nabla^{-1}P_{k_2}Q_{<k_2+\mu k_1}\phi_2||_{L_t^4L_x^\infty}||P_{k_3}\phi_3||_{L_t^4L_x^p}\\&\leq C_p 2^{(\frac{3p-12}{4p}-\frac{1}{2})k_1}2^{\frac{k_2}{4}}\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]}.\end{aligned}$$

Now choose p close enough to 4.

(III): $P_0[\nabla^{-1}(\nabla^{-1}\Box P_{k_2}Q_{<k_2+\mu k_1}\phi_2 P_{k_1}Q_{<k_2}\phi_1)P_{k_3}\phi_3]$. Choose $p>4$ and M with the property $\frac{2}{p}+\frac{1}{M}=\frac{1}{2}$. Then we estimate

$$
\begin{aligned}
&||P_0[\nabla^{-1}(\nabla^{-1}\Box P_{k_2}Q_{<k_2+\mu k_1}\phi_2 P_{k_1}Q_{<k_2}\phi_1)P_{k_3}\phi_3]||_{L_t^1 L_x^2}\\
&\leq C2^{-k_1}||\nabla^{-1}\Box P_{k_2}Q_{<k_2+\mu k_1}\phi_2||_{L_t^2 L_x^M}||P_{k_1}Q_{<k_2}\phi_1||_{L_t^4 L_x^p}||P_{k_3}\phi_3||_{L_t^4 L_x^p}\\
&\leq C_p 2^{(\frac{3}{2}-\frac{3}{M})k_2}2^{\frac{3\mu}{2}k_1}2^{k_1(\frac{3p-12}{6}-1)}\prod_i ||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

The desired estimate follows again easily from this.

(B): *Low-High interactions:* $P_0[\nabla^{-1}P_{<15}(\partial_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\phi_2)P_{k_3}\phi_3]$, $-10<k_3\leq 10$.

(B.1): High-High/Low-High interactions within (,): $k_1\leq k_2+O(1)$. We use lemma 2.2 as well as theorem 3.3:

$$
\begin{aligned}
&||P_0[\nabla^{-1}P_{<15}(\partial_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\phi_2)P_{k_3}\phi_3]||_{N[0]}\\
&\leq \sum_{k<k_2+O(1)}||P_0[\nabla^{-1}P_k(\partial_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\phi_2)P_{k_3}\phi_3]||_{N[0]}\\
&\leq C\sum_{k<k_2+O(1)}2^{\delta(k-k_2)}2^{k_1-k_2}\prod_{i=1}^3 ||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

This yields the claim of the theorem.

(B.2): $k_2<<k_1$. We need to reduce the modulations:

(B.2.1): $P_{k_2}\phi_2$ at modulation $\geq 2^{k_2}$: let $\frac{2}{p}+\frac{1}{M}=\frac{1}{2}$.

$$
\begin{aligned}
&||P_0[\nabla^{-1}P_{<15}(\partial_\nu P_{k_1}\phi_1 R^\nu P_{k_2}Q_{\geq k_2}\phi_2)P_{k_3}\phi_3]||_{L_t^1 L_x^2}\\
&\leq C2^{-k_1}||R^\nu P_{k_2}Q_{\geq k_2}\phi_2||_{L_t^2 L_x^M}||\partial_\nu P_{k_1}\phi_1||_{L_t^4 L_x^p}||P_{k_3}\phi_3||_{L_t^4 L_x^p}\\
&\leq C2^{\frac{k_2}{2+}}\prod_i ||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

(B.2.2): $P_{k_2}\phi_2$ at modulation $< 2^{k_2}$, $P_{k_1}\phi_1$ at modulation $\geq 2^{k_2}$. This is handled like the previous case.

(B.2.3): Expand the null-form. The following terms need to be estimated:

(I): $P_0[\nabla^{-1}\Box(P_{k_1}Q_{<k_2}\phi_1\nabla^{-1}P_{k_2}Q_{<k_2}\phi_2)P_{k_3}\phi_3]$. Use lemma 2.2 as well as theorem 3.2:

$$
\begin{aligned}
&||P_0[\nabla^{-1}\Box(P_{k_1}Q_{<k_2}\phi_1\nabla^{-1}P_{k_2}Q_{<k_2}\phi_2)P_{k_3}\phi_3]||_{N[0]}\\
&\leq \sum_{j<k_2+O(1)} ||P_0[\nabla^{-1}\Box Q_j(P_{k_1}Q_{<k_2}\phi_1\nabla^{-1}P_{k_2}Q_{<k_2}\phi_2)P_{k_3}\phi_3]||_{N[0]}\\
&\leq C\sum_{j<k_2+O(1)} 2^{\delta(j-k_1)}||\nabla^{-1}\Box Q_j(P_{k_1}Q_{<k_2}\phi_1\nabla^{-1}P_{k_2}Q_{<k_2}\phi_2)||_{\dot{X}^{\frac{1}{2},-\frac{1}{2},\infty}_{k_1+O(1)}}||P_{k_3}\phi_3||_{S[k_3]}\\
&\leq C\sum_{j<k_2+O(1)} 2^{\delta(j-k_1)}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

(II): $P_0[\nabla^{-1}(P_{k_1}Q_{<k_2}\Box\phi_1\nabla^{-1}P_{k_2}Q_{<k_2}\phi_2)P_{k_3}\phi_3]$. Use Strichartz type norms, $p > 4$:

$$
\begin{aligned}
&||P_0[\nabla^{-1}(P_{k_1}Q_{<k_2}\Box\phi_1\nabla^{-1}P_{k_2}Q_{<k_2}\phi_2)P_{k_3}\phi_3]||_{L_t^1L_x^2}\\
&\leq C2^{-k_1}||P_{k_1}Q_{<k_2}\Box\phi_1||_{L_t^2L_x^{4-}}||\nabla^{-1}P_{k_2}Q_{<k_2}\phi_2||_{L_t^4L_x^\infty}||P_{k_3}\phi_3|||_{L_t^4L_x^p}\\
&\leq C2^{\frac{3k_1}{4+}}2^{-\frac{k_1}{2}}2^{\frac{k_2}{4}}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

(III): $P_0[\nabla^{-1}(P_{k_1}Q_{<k_2}\phi_1\nabla^{-1}P_{k_2}Q_{<k_2}\Box\phi_2)P_{k_3}\phi_3]$. This is more of the same and left out.

(C): *High-Low interactions:* $P_0[\nabla^{-1}P_{>-15}(\partial_\nu P_{k_1}\phi_1 R^\nu P_{k_2}\phi_2)P_{k_3}\phi_3]$, $k_3 \leq -10$. They are easier to treat than the preceding since ∇^{-1} falls on a high frequency term. We leave this for the reader. ■

The next theorem is the crucial trilinear null-form estimate needed for the treatment of Wave Maps in $3+1$ dimensions. It is significantly more sophisticated than the preceding estimate, on account of the more delicate nature of *low-high interactions.*

Theorem 4.2. *Let ϕ_i, $i = 1, 2, 3$, be Schwartz functions on $\mathbf{R}^{3+1}$. Then the following inequality holds true:*

$$||P_0[\sum_{j=1}^{3}\triangle^{-1}\partial_j(R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)\partial^\nu P_{k_3}\phi_3]||_{N[0]}$$
$$\leq C2^{-\delta_1|k_1-k_2|}2^{\delta_2\min\{k_3-\max\{k_1,k_2\},0\}}2^{-\delta_3|k_3|}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}$$

for appropriate constants $\delta_i > 0$, $i = 1,2,3$.

Before beginning with the proof, we state the following elementary lemma:

Lemma 4.3. *Let* f, g, h *be Schwartz functions. Then we have*

$$\begin{aligned}
&2\sum_{j=1}^{3}\triangle^{-1}\partial_j[R_\nu f R_j g - R_j f R_\nu g]\partial^\nu h\\
&\sum_{j=1}^{3}\Box[\triangle^{-1}\partial_j[\nabla^{-1}fR_jg]h] - \sum_{j=1}^{3}\Box\triangle^{-1}\partial_j[\nabla^{-1}fR_jg]h\\
&-\sum_{j=1}^{3}\triangle^{-1}\partial_j[\nabla^{-1}fR_jg]\Box h - \nabla^{-1}f\Box((\nabla^{-1}g)h)\\
&+\nabla^{-1}f\Box(\nabla^{-1}g)h + \nabla^{-1}f(\nabla^{-1}g)\Box h.
\end{aligned} \tag{15}$$

Proof : Use the identities

$$R_\nu f R_j g - R_j f R_\nu g = \partial_\nu(\sqrt{-\triangle}^{-1} f R_j g) - \partial_j(\sqrt{-\triangle}^{-1} f R_\nu g),$$

$$2\partial_\nu f\partial^\nu g = \Box(fg) - \Box f g - f\Box g.$$

■

Now we begin with the proof of the theorem:

Proof : As usual, the proof is of a fairly mechanical nature. The idea is as in the proof of previous estimates to *reduce the modulations of the inputs sufficiently in order to be able to take advantage of the inherent null-structure* (15). We distinguish between **Low-High, High-High** as well as **High-Low** interactions.

(A): Low-High interactions. $k_3 \in [-10, 10]$. It is the most difficult case, on account of the fact that the ∇^{-1} operator falls on a low-frequency term. We further distinguish between *Low-High, High-Low, High-High interactions* within (,):

(A.1): $k_1 < k_2 - 10$. We commence by reducing the *output* as well as the *third input* $P_{k_3}\partial^\nu\phi_3$ to modulation $< 2^{k_2-100}$:

(A.1.1): *Output at modulation* $\geq 2^{k_2-100}$. Use theorem 3.2:

$$||P_0Q_{\geq k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j(R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)\partial^\nu P_{k_3}\phi_3]||_{N[0]}$$

$$\leq ||P_0Q_{\geq k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j(R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)\partial^\nu P_{k_3}\phi_3]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}}$$

$$\leq C2^{-\frac{k_2}{2}}2^{-k_2}\sum_{j=1}^{3}||R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2||_{L_t^2 L_x^\infty}||\partial^\nu P_{k_3}\phi_3||_{L_t^\infty L_x^2}$$

$$\leq C2^{-\frac{|k_1-k_2|}{2}}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.$$

(A.1.2): *Output at modulation* $< 2^{k_2-100}$*,* $P_{k_3}\partial^\nu\phi_3$ *at modulation* $\geq 2^{k_2-100}$*:* As in the preceding proofs, we usually place inputs which are 'far away from the light cone' into $L_t^2L_x^2$, using the $\dot{X}^{\frac{1}{2},\frac{1}{2},\infty}$ component of $S[k]$. Then we use theorem 3.1 or theorem 3.2:

$$||P_0Q_{<k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j(R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)$$
$$\partial^\nu P_{k_3}Q_{\geq k_2-100}\phi_3]||_{N[0]}$$

$$\leq ||P_0Q_{<k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j(R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)$$
$$\partial^\nu P_{k_3}Q_{\geq k_2-100}\phi_3]||_{L_t^1\dot{H}^{-\frac{1}{2}}}$$

$$\leq C2^{-k_2}\sum_{j=1}^{3}||R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2||_{L_t^2L_x^\infty}$$
$$||\partial^\nu P_{k_3}Q_{\geq k_2-100}\phi_3||_{L_t^2L_x^2}$$

$$\leq C2^{-\frac{|k_1-k_2|}{2}}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.$$

(A.1.3): *Output and third input at modulation* $< 2^{k_2-100}$*. Part of null-structure becomes useful.*

We now use the simple identity

$$
\begin{aligned}
&2\sum_{j=1}^{3}\triangle^{-1}\partial_j[R_\nu f R_j g - R_j f R_\nu g]\partial^\nu h\\
&\sum_{j=1}^{3}\Box[\triangle^{-1}\partial_j[\nabla^{-1}fR_jg]h] - \sum_{j=1}^{3}\Box\triangle^{-1}\partial_j[\nabla^{-1}fR_jg]h\\
&-\sum_{j=1}^{3}\triangle^{-1}\partial_j[\nabla^{-1}fR_jg]\Box h - 2R_\nu f\nabla^{-1}g\partial^\nu h.
\end{aligned}
$$

We substitute the appropriately microlocalized inputs and begin by estimating the first three terms on the right-hand side of the preceding expansion. The reductions of the modulations effected thus far turn out to be sufficient for that purpose. The fourth term on the other hand will turn out to be more complicated and requires further modulation reductions before we can take advantage of its null-structure:

(A.1.3.a):

$$
\begin{aligned}
&||\Box P_0 Q_{<k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}Q_{<k_2-100}\phi_3]||_{N[0]}\\
&\leq\sum_{j=1}^{3}||\Box P_0 Q_{<k_2-100}[\triangle^{-1}\partial_j(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}Q_{<k_2-100}\phi_3]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}}\\
&\leq C2^{\frac{k_2}{2}}||\triangle^{-1}\partial_j(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)||_{L_t^2L_x^\infty}||P_{k_3}Q_{<k_2-100}\phi_3||_{L_t^\infty L_x^2}\\
&\leq C2^{-\frac{|k_1-k_2|}{2}}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

(A.1.3.b):

$$
\begin{aligned}
&||P_0 Q_{<k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j\Box(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}Q_{<k_2-100}\phi_3]||_{N[0]}\\
&\leq\sum_{r<k_2+O(1)}||P_0 Q_{<k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j\Box Q_r(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}Q_{<k_2-100}\phi_3]||_{N[0]}.
\end{aligned}
$$

Unfortunately, the operator $P_0Q_{<k_2-100}$ is *not disposable*[12], and we have to get rid of it before we can estimate the preceding term. We simply decompose

[12]Recall that this means that it is not given by convolution with a kernel of L^1-mass $< O(1)$.

$$||P_0Q_{<k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j\Box Q_r(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}Q_{<k_2-100}\phi_3]||_{N[0]}$$
$$\leq ||P_0[\sum_{j=1}^{3}\triangle^{-1}\partial_j\Box Q_r(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}\phi_3]||_{N[0]}$$
$$+ ||P_0Q_{\geq k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j\Box Q_r(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}\phi_3]||_{N[0]}$$
$$+ ||P_0Q_{<k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j\Box Q_r(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}Q_{\geq k_2-100}\phi_3]||_{N[0]}.$$

The 2nd and third summand on the right-hand side of the immediately preceding equality are of course estimated like **(A.1.1)**, **(A.1.2)**, hence we need to focus on the first. Use lemma 2.1, as well as theorem 3.2:

$$\sum_{r<k_2+O(1)}||P_0[\sum_{j=1}^{3}\triangle^{-1}\partial_j\Box Q_r(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}\phi_3]||_{N[0]}$$
$$\leq C\sum_{r<k_2+O(1)}\sum_{j=1}^{3}2^{\delta(r-k_2)}||\Box Q_r(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)||_{\dot{X}_{k_2}^{-\frac{1}{2},-\frac{1}{2},\infty}}||P_{k_3}\phi_3||_{S[k_3]}$$
$$\leq C\sum_{r<k_2+O(1)}2^{\delta(r-k_2)}2^{k_1-k_2}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.$$

This is of the desired form.

(A.1.3.c):

$$||P_0Q_{<k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}Q_{<k_2-100}\Box\phi_3]||_{N[0]}$$
$$\leq ||P_0Q_{<k_2-100}[\sum_{j=1}^{3}\triangle^{-1}\partial_j(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)P_{k_3}Q_{<k_2-100}\Box\phi_3]||_{L_t^1\dot{H}^{-\frac{1}{2}}}$$
$$\leq C||\sum_{j=1}^{3}\triangle^{-1}\partial_j(P_{k_1}R_j\phi_1\nabla^{-1}P_{k_2}\phi_2)||_{L_t^2L_x^\infty}||P_{k_3}Q_{<k_2-100}\Box\phi_3]||_{L_t^2L_x^2}$$
$$\leq C2^{\frac{k_1-k_2}{2}}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.$$

(A.1.3.d):

$$P_0Q_{<k_2-100}[R_\nu P_{k_1}\phi_1\nabla^{-1}P_{k_2}\phi_2\partial^\nu P_{k_3}Q_{<k_2-100}\phi_3].$$

This term is more complicated since the modulation reductions(i.e. the operators $Q_{<k_2-100}$) are not sufficient to obtain an exponential gain in the difference k_1-k_2. We shall indeed abolish these operators and state what we want to prove as a separate lemma:

Lemma 4.4. *Let ϕ_i, $i=1, 2, 3$, be Schwartz functions. Also, let k_i, $i=1, 2, 3$, be as in the preceding. Then the following inequality holds:*

$$||P_0[R_\nu P_{k_1}\phi_1\nabla^{-1}P_{k_2}\phi_2\partial^\nu P_{k_3}\phi_3]||_{N[0]}\leq C2^{\delta(k_1-k_2)}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}$$

for suitable $\delta>0$.

Remark: The case **(A.1.3.d)** is then handled by means of the simple observation

$$\begin{aligned}&||P_0Q_{\geq k_2-100}[R_\nu P_{k_1}\phi_1\nabla^{-1}P_{k_2}\phi_2\partial^\nu P_{k_3}Q_{<k_2-100}\phi_3]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}}\\&\leq C2^{-\frac{k_2}{2}}||R_\nu P_{k_1}\phi_1||_{L_t^4L_x^\infty}||\nabla^{-1}P_{k_2}\phi_2||_{L_t^4L_x^\infty}||\partial^\nu P_{k_3}Q_{<k_2-100}\phi_3||_{L_t^\infty L_x^2}\\&\leq C2^{\frac{3(k_1-k_2)}{4}}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]},\end{aligned}$$

as well as a similar inequality getting rid of the $Q_{<k_2-100}$ in front of the third input.

Proof of the lemma: We need to reduce the modulations of $P_{k_1}\psi_1$ as well as $P_{k_3}\phi_3$ to size $<2^{k_1+O(1)}$ before being able to take advantage of the inherent Q_0 null-structure. We shall achieve this in a stepwise fashion:

(a): *$P_{k_1}\phi_1$ at modulation in the range $[2^{k_1+100}, 2^{k_2+100}]$, $P_{k_3}\phi_3$ at modulation $<2^{-100}$modulation$(P_{k_1}\phi_1)$.*

We represent this case as follows:

$$\begin{aligned}&\sum_{k_2+100\geq r\geq k_1+100}P_0[R_\nu P_{k_1}Q_r\phi_1\nabla^{-1}P_{k_2}\phi_2\partial^\nu P_{k_3}Q_{<r-100}\phi_3]\\&=\sum_{k_2+100\geq r\geq k_1+100}P_0[Q_{r+O(1)}(R_\nu P_{k_1}Q_r\phi_1\partial^\nu P_{k_3}Q_{<r-100}\phi_3)\nabla^{-1}P_{k_2}\phi_2].\end{aligned}$$

The last term calls for application of lemma 2.1:

$$\begin{aligned}
&|| \sum_{k_2+100 \geq r \geq k_1+100} P_0[Q_{r+O(1)}(R_\nu P_{k_1} Q_r \phi_1 \\
&\qquad \partial^\nu P_{k_3} Q_{<r-100}\phi_3)\nabla^{-1} P_{k_2}\phi_2]||_{N[0]} \\
&\leq \sum_{k_2+100 \geq r \geq k_1+100} C2^{\delta(r-k_2)} ||Q_{r+O(1)}(R_\nu P_{k_1} Q_r \phi_1 \\
&\qquad \partial^\nu P_{k_3} Q_{<r-100}\phi_3)||_{\dot{X}_{k_3}^{\frac{1}{2},-\frac{1}{2},\infty}} ||P_{k_2}\phi_2||_{S[k_2]} \\
&\leq C \sum_{k_2+100 \geq r \geq k_1+100} 2^{\delta(r-k_2)} 2^{-\frac{r}{2}} ||R_\nu P_{k_1} Q_r \phi_1||_{L_t^2 L_x^\infty} \\
&\qquad ||\partial^\nu P_{k_3} Q_{<r-100}\phi_3||_{L_t^\infty L_x^2} ||P_{k_2}\phi_2||_{S[k_2]} \\
&\leq C \sum_{k_2+100 \geq r \geq k_1+100} 2^{k_1-r} 2^{\delta(r-k_2)} \prod_{i=1}^{3} ||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}$$

The last sum can be easily carried out to obtain the desired estimate.

(b): *$P_{k_1}\phi_1$ has modulation in the range $[2^{k_1+100}, 2^{k_2+100}]$, $P_{k_3}\phi_3$ has modulation in the range $[2^{-100} modulation(P_{k_1}\phi_1), 2^{k_2+100}]$, $P_{k_2}\phi_2$ at modulation $< 2^{-100} modulation(P_{k_1}\phi_1)$.*

We group the terms differently for application of lemma 2.1:

$$\begin{aligned}
&|| \sum_{k_2+100 \geq r \geq k_1+100} \sum_{k_2+100 \geq a \geq r-100} P_0[R_\nu P_{k_1} Q_r \phi_1 \nabla^{-1} P_{k_2} Q_{<r-100}\phi_2 \partial^\nu P_{k_3} Q_a \phi_3]||_{N[0]} \\
&\leq \sum_{k_2+100 \geq r \geq k_1+100} \sum_{k_2+100 \geq a \geq r-100} C2^{\delta(a-k_2)} ||Q_{r+O(1)} \nabla_x [R_\nu P_{k_1} Q_r \phi_1 \\
&\qquad \nabla^{-1} P_{k_2} Q_{<r-100}\phi_2]||_{S[k_2]} ||P_{k_3} Q_a \partial^\nu \phi_3||_{\dot{X}_{k_3}^{\frac{1}{2},-\frac{1}{2},\infty}}.
\end{aligned}$$

Now we use the basic inequality(cf. section 2)

$$||P_k Q_{<k}\phi||_{S[k]} \leq C ||P_k \phi||_{\dot{X}_k^{\frac{1}{2},\frac{1}{2},1}}.$$

Therefore

$$\begin{aligned}
&||Q_{r+O(1)} \nabla_x [R_\nu P_{k_1} Q_r \phi_1 \nabla^{-1} P_{k_2} Q_{<r-100}\phi_2]||_{S[k_2]} \\
&\leq C 2^{\frac{r}{2}} 2^{\frac{3k_2}{2}} ||R_\nu P_{k_1} Q_r \phi_1||_{L_t^2 L_x^\infty} ||P_{k_2} Q_{<r-100} \nabla^{-1}\phi_2||_{L_t^\infty L_x^2} \\
&\leq C 2^{k_1} ||P_{k_1}\phi_1||_{S[k_1]} ||P_{k_2}\phi_2||_{S[k_2]}.
\end{aligned}$$

We conclude that

$$||\sum_{k_2+100\geq r\geq k_1+100}\sum_{k_2+100\geq a\geq r-100}P_0[R_\nu P_{k_1}Q_r\phi_1$$
$$\nabla^{-1}P_{k_2}Q_{<r-100}\phi_2\partial^\nu P_{k_3}Q_a\phi_3]||_{N[0]}$$
$$\leq |\sum_{k_2+100\geq r\geq k_1+100}\sum_{k_2+100\geq a\geq r-100}C2^{k_1-a}2^{\delta(a-k_2)}\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]}$$
$$\leq C2^{\frac{\delta}{2}(k_1-k_2)}\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]}.$$

(c): *$P_{k_1}\phi_1$ has modulation in the range $[2^{k_1+100},2^{k_2+100}]$, $P_{k_3}\phi_3$ has modulation in the range $[2^{-100}modulation(P_{k_1}\phi_1),2^{k_2+100}]$, $P_{k_2}\phi_2$ at modulation $\geq 2^{-100}modulation(P_{k_1}\phi_1)$.*

We estimate this contribution as follows, by means of the improved Bernstein's inequality (8):

$$||\sum_{k_2+100\geq r\geq k_1+100}\sum_{a\geq r-100}P_0[R_\nu P_{k_1}Q_r\phi_1\nabla^{-1}P_{k_2}Q_a\phi_2\partial^\nu P_{k_3}Q_{k_2+100\geq .\geq r-100}\phi_3]||_{N[0]}$$
$$\leq \sum_{k_2+100\geq r\geq k_1+100}\sum_{a\geq r-100}C||R_\nu P_{k_1}Q_r\phi_1||_{L_t^2L_x^\infty}||\nabla^{-1}P_{k_2}Q_a\phi_2||_{L_t^2L_x^\infty}$$
$$||\partial^\nu P_{k_3}Q_{k_2+100\geq .\geq r-100}\phi_3||_{L_t^\infty L_x^2}$$
$$\leq C\sum_{k_2+100\geq r\geq k_1+100}\sum_{a\geq r-100}2^{-\frac{r}{2}}2^{-\frac{k_1}{2}}2^{\frac{3k_1}{2}}2^{\min\{\frac{a-k_2}{2+},0\}}2^{-\frac{a}{2}}2^{-\frac{k_2}{2}}2^{\frac{3k_2}{2}}2^{-k_2}$$
$$\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]}$$
$$\leq C2^{\frac{k_1-k_2}{2+}}\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]}.$$

(d): *$P_{k_1}\phi_1$ at modulation in the range $[2^{k_1+100},2^{k_2+100}]$, $P_{k_3}\phi_3$ at modulation $>2^{k_2+100}$.* We estimate

$$||\sum_{k_2+100\geq r\geq k_1+100}P_0[R_\nu P_{k_1}Q_r\phi_1\nabla^{-1}P_{k_2}\phi_2\partial^\nu P_{k_3}Q_{>k_2+100}\phi_3]||_{L_t^1L_x^2}$$
$$\leq \sum_{k_2+100\geq r\geq k_1+100}C||R_\nu P_{k_1}Q_r\phi_1||_{L_t^2L_x^\infty}||\nabla^{-1}P_{k_2}\phi_2||_{L_t^\infty L_x^\infty}||\partial^\nu P_{k_3}Q_{>k_2+100}\phi_3||_{L_t^2L_x^2}$$
$$\leq \sum_{k_2+100\geq r\geq k_1+100}C2^{k_1-\frac{r}{2}-\frac{k_2}{2}}\prod_i||P_{k_i}\phi_i||_{S[k_i]}.$$

Of course this yields the required inequality.

(e): *$P_{k_1}\phi_1$ at modulation $> 2^{k_2+100}$, at least one of $P_{k_2}\nabla^{-1}\phi_2$, $P_{k_3}\phi_3$ at modulation $> 2^{k_2-100}$.* This is treated like the preceding case and left out.

(f): *$P_{k_1}\phi_1$ at modulation $> 2^{k_2+100}$, both $P_{k_2}\nabla^{-1}\phi_2$ and $P_{k_3}\phi_3$ at modulation $< 2^{k_2-100}$.* We have the identity

$$\begin{aligned}&\sum_{r>k_2+100} P_0[R_\nu P_{k_1}Q_r\phi_1\nabla^{-1}P_{k_2}Q_{<k_2-100}\phi_2\partial^\nu P_{k_3}Q_{<k_2-100}\phi_3]\\ &=\sum_{r>k_2+100} P_0Q_{r+O(1)}[R_\nu P_{k_1}Q_r\phi_1\nabla^{-1}P_{k_2}Q_{<k_2-100}\phi_2\partial^\nu P_{k_3}Q_{<k_2-100}\phi_3].\end{aligned}$$

Therefore, we can estimate

$$\begin{aligned}&||\sum_{r>k_2+100} P_0[R_\nu P_{k_1}Q_r\phi_1\nabla^{-1}P_{k_2}Q_{<k_2-100}\phi_2\partial^\nu P_{k_3}Q_{<k_2-100}\phi_3]||_{N[0]}\\ &\leq C\sum_{r>k_2+100} 2^{-\frac{r}{2}}||R_\nu P_{k_1}Q_r\phi_1||_{L_t^2L_x^\infty}||\nabla^{-1}P_{k_2}Q_{<k_2-100}\phi_2||_{L_t^\infty L_x^\infty}\\ &\qquad\qquad ||\partial^\nu P_{k_3}Q_{<k_2-100}\phi_3||_{L_t^\infty L_x^2}\\ &\leq C2^{k_1-k_2}\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]}.\end{aligned}$$

(g): *$P_{k_1}\phi_1$ at modulation $< 2^{k_1+100}$.* Before expanding the null-structure, we also need to reduce the high-frequency term $P_{k_3}\phi_3$ to modulation $< 2^{k_1}$:

$$\begin{aligned}&||P_0[R_\nu P_{k_1}Q_{<k_1+100}\phi_1\nabla^{-1}P_{k_2}\phi_2\partial^\nu P_{k_3}Q_{\geq k_1}\phi_1]||_{N[0]}\\ &\leq C||R_\nu P_{k_1}Q_{<k_1+100}\phi_1||_{L_t^4L_x^\infty}||\nabla^{-1}P_{k_2}\phi_2||_{L_t^4L_x^\infty}||\partial^\nu P_{k_3}Q_{\geq k_1}\phi_1||_{L_t^2L_x^2}\\ &\leq C2^{\frac{k_1-k_2}{4}}\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]}.\end{aligned}$$

We can now expand the null-structure. The following three terms need to be estimated:

(g.1): $P_0[\Box(\nabla^{-1}P_{k_1}Q_{<k_1+100}\phi_1P_{k_3}Q_{<k_1}\phi_3)\nabla^{-1}P_{k_2}\phi_2]$: Use lemma 2.1 as well as theorem 3.2.

$$
\begin{aligned}
&||P_0[\Box(\nabla^{-1}P_{k_1}Q_{<k_1+100}\phi_1 P_{k_3}Q_{<k_1}\phi_3)\nabla^{-1}P_{k_2}\phi_2]||_{N[0]}\\
&= ||P_0[\Box Q_{<k_1+O(1)}(\nabla^{-1}P_{k_1}Q_{<k_1+100}\phi_1 P_{k_3}Q_{<k_1}\phi_3)\nabla^{-1}P_{k_2}\phi_2]||_{N[0]}\\
&\leq C\sum_{j<k_1+O(1)}||\Box Q_j(\nabla^{-1}P_{k_1}Q_{<k_1+100}\phi_1 P_{k_3}Q_{<k_1}\phi_3)||_{\dot{X}_{k_3}^{\frac{1}{2},-\frac{1}{2},\infty}}||P_{k_2}\phi_2||_{S[k_2]}\\
&\leq C\sum_{j<k_1+O(1)}2^{\delta(j-k_2)}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

Of course this is acceptable.

(g.2): $P_0[\nabla^{-1}P_{k_1}Q_{<k_1+100}\phi_1 P_{k_3}Q_{<k_1}\Box\phi_3\nabla^{-1}P_{k_2}\phi_2]$: Use theorem 3.2:

$$
\begin{aligned}
&||P_0[\nabla^{-1}P_{k_1}Q_{<k_1+100}\phi_1 P_{k_3}Q_{<k_1}\Box\phi_3\nabla^{-1}P_{k_2}\phi_2]||_{N[0]}\\
&\leq C||\nabla^{-1}P_{k_1}Q_{<k_1+100}\phi_1||_{L_t^4L_x^\infty}||\nabla^{-1}P_{k_2}\phi_2||_{L_t^4L_x^\infty}||P_{k_3}Q_{<k_1}\Box\phi_3||_{L_t^2L_x^2}\\
&\leq C2^{\frac{k_1-k_2}{4}}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

(g.3): $P_0[\nabla^{-1}P_{k_1}Q_{<k_1+100}\Box\phi_1 P_{k_3}Q_{<k_1}\phi_3\nabla^{-1}P_{k_2}\phi_2]$: Use theorem 3.2 and Bernstein's inequality (7):

$$
\begin{aligned}
&||P_0[\nabla^{-1}P_{k_1}Q_{<k_1+100}\Box\phi_1 P_{k_3}Q_{<k_1}\phi_3\nabla^{-1}P_{k_2}\phi_2]||_{N[0]}\\
&\leq C||\nabla^{-1}P_{k_1}Q_{<k_1+100}\Box\phi_1||_{L_t^2L_x^{4-}}||\nabla^{-1}P_{k_2}\phi_2||_{L_t^4L_x^\infty}||P_{k_3}Q_{<k_1}\phi_3||_{L_t^4L_x^{4+}}\\
&\leq C2^{\frac{3k_1}{4+}}2^{-\frac{k_2}{4}}\prod_{i=1}^{3}||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

This completes the proof of the lemma, and hence of Case **(A.1)**. ■

(A.2): $|k_1-k_2|<O(1)$. Thus this case corresponds to a high-high interaction within $(,)$, and can be rewritten as

$$
\sum_{k\leq\min\{k_1+O(1),O(1)\}}P_0[\sum_{j=1}^{3}\triangle^{-1}\partial_j P_k(R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)P_{k_3}\partial^\nu\phi_3].
$$

We want to proceed in analogy to the case **(A.1)**, by first reducing output and input $P_{k_3}\psi_3$ to small modulation (in this case modulation $<2^{k-100}$), where $k\leq\min\{k_1+O(1),O(1)\}$ is held fixed. Since we are eventually summing over k, we want to obtain an exponential gain in the difference $k-k_1$.

(A.2.1): *Output has modulation in the range* $[2^{k-100}, 2^{k_1+100}]$, $P_{k_3}\phi_3$ *at modulation* $< 2^k$. Freeze the modulation of the output to dyadic size 2^l, $l \in [k-100, k_1+100]$. Note that our assumptions force $(,)$ to be at modulation $< 2^{l+O(1)}$. We shall exploit the crude identity

$$R_\nu \phi R_j \psi - R_j \phi R_\nu \psi = \nabla_{x,t} \nabla^{-1}(\nabla^{-1}\phi\psi).$$

Therefore, we have

$$\begin{aligned}
&|| \sum_{k-100 \le l \le k_1+100} P_0 Q_l [\sum_{j=1}^{3} P_k \triangle^{-1} \partial_j (R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 \\
&\qquad\qquad - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2) P_{k_3} Q_{<k} \partial^\nu \phi_3]||_{N[0]} \\
&\le || \sum_{k-100 \le l \le k_1+100} P_0 Q_l [\sum_{j=1}^{3} P_k \triangle^{-1} \partial_j Q_{<l+O(1)} (R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 \\
&\qquad\qquad - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2) P_{k_3} Q_{<k} \partial^\nu \phi_3]||_{N[0]} \\
&\le C \sum_{k-100 \le l \le k_1+100} 2^{-\frac{l}{2}} 2^{l-k} ||\nabla_{x,t} \nabla^{-1} P_{k_1}\phi_1 \nabla^{-1} P_{k_2}\phi_2||_{L_t^2 L_x^\infty} ||P_{k_3} Q_{<k} \partial^\nu \phi_3||_{L_t^\infty L_x^2} \\
&\le C 2^{\frac{k-k_1}{2}} \prod_{i=1}^{3} ||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}$$

(A.2.2): *Output has modulation in the range* $[2^{k-100}, 2^{k_1+100}]$, $P_{k_3}\phi_3$ *at modulation* $\ge 2^k$. This is easier and estimated by placing $(,)$ into $L_t^\infty L_x^\infty$ while placing $P_{k_3} Q_{\ge k}\phi_3$ into $L_t^2 L_x^2$:

$$\begin{aligned}
&|| \sum_{k-100 \le l \le k_1+100} P_0 Q_l [\sum_{j=1}^{3} P_k \triangle^{-1} \partial_j (R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 \\
&\qquad\qquad - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2) P_{k_3} Q_{\ge k} \partial^\nu \phi_3]||_{N[0]} \\
&\le C \sum_{k-100 \le l \le k_1+100} 2^{-\frac{l}{2}} || \sum_{j=1}^{3} P_k \triangle^{-1} \partial_j (R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 \\
&\qquad\qquad - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)||_{L_t^\infty L_x^\infty} ||P_{k_3} Q_{\ge k} \partial^\nu \phi_3||_{L_t^2 L_x^2} \\
&\le C 2^{k-k_1} \prod_{i=1}^{3} ||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}$$

(A.2.3): *Output at modulation* $> 2^{k_1+100}$, $P_{k_3}\phi_3$ *at modulation* $< 2^{-100}$*modulation*(*output*). Let the modulation be frozen at dyadic value 2^l, $l > k_1 + 100$. Then at least one input of $(,)$ needs to be at modulation $\ge 2^{l-10}$. One places this input into $L_t^2 L_x^2$, and the whole output into $\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}$. For example, we have

$$||\sum_{l>k_1+100}P_0Q_l[\sum_{j=1}^3\triangle^{-1}\partial_j(R_\nu P_{k_1}Q_{\geq l-10}\phi_1R_jP_{k_2}\phi_2)P_{k_3}Q_{<l-100}\phi_3]||_{N[0]}$$
$$\leq\sum_{l>k_1+100}\sum_{j=1}^3 2^{-\frac{l}{2}}2^{2k-\frac{l}{2}-k_1}\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]}\leq C2^{2(k-k_1)}\prod_{i=1}^3||P_{k_i}\phi_i||_{S[k_i]}.$$

(A.2.4.): *Output at modulation* $>2^{k_1+100}$*,* $P_{k_3}\phi_3$ *at modulation* $\geq 2^{-100}$*modulation(output).* Place the output, assumed at modulation 2^l, $l>k_1+100$, into $\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}$, and $P_{k_3}Q_{\geq l-100}\phi_3$ into $L_t^2L_x^2$.

(A.2.5): *Output at modulation* $<2^{k-100}$*,* $P_{k_3}\phi_3$ *at modulation* $>2^{k+100}$. Assume $P_{k_3}\phi_3$ to be at modulation $\sim 2^l$, $l>k+100$. This forces $(,)$ to be at modulation $\sim 2^{l+O(1)}$. Now one uses the simple inequality

$$||P_kQ_l\nabla^{-1}(R_\nu P_{k_1}\phi R_jP_{k_2}\phi_2-R_jP_{k_1}\phi R_\nu P_{k_2}\phi_2)||_{L_t^2L_x^\infty}$$
$$\leq C2^{\frac{k}{2}}2^{\frac{k-l}{2}}2^{\frac{k-k_1}{2}}\prod_{i=1,2}||P_{k_i}\phi_i||_{S[k_i]}.$$

(A.2.6): *Output at modulation* $<2^{k-100}$*,* $P_{k_3}\phi_3$ *at modulation* $<2^{k+100}$. This entails that $(,)$ is at modulation $<2^{k+O(1)}$ as well. One can now expand the null-structure (15) as in the preceding case **(A.1)**, and concludes in the same way. This finishes **(A)**.

(B): High-High interactions. $k_3>10$. We shall again take advantage of the null-form, but in a more crude fashion than before. In particular, we shall reduce this case to an already known trilinear estimate.

(B.1): $P_{k_3}\partial^\nu\phi_3$ *at modulation* $\geq 2^{k_3}$. Simply place this input into $L_t^2L_x^2$, and use theorem 3.2.

(B.2): *Output at modulation* >1*,* $P_{k_3}\partial^\nu\phi_3$ *at modulation* $<2^{k_3}$. Place the output into $\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}$, and use theorem 3.2.

(B.3): *Output at modulation* ≤ 1*,* $P_{k_3}\partial^\nu\phi_3$ *at modulation* $<2^{k_3}$. Assume w. l. o. g. that $k_1\geq k_2$, whence $k_3\leq k_1+O(1)$. We use the simple inequality

$$||P_0Q_{<0}[\sum_{j=1}^{3}\triangle^{-1}\partial_j(R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)\partial^\nu P_{k_3}Q_{<k_3}\phi_3]||_{N[0]}$$
$$\leq ||P_0Q_{<0}[\sum_{j=1}^{3}\triangle^{-1}\partial_j\partial_\nu(R_j P_{k_2}\phi_2\nabla^{-1}P_{k_1}\phi_1)\partial^\nu P_{k_3}Q_{<k_3}\phi_3]||_{N[0]}$$
$$+ ||P_0Q_{<0}[\sum_{j=1}^{3}R_\nu P_{k_2}\phi_2\nabla^{-1}P_{k_1}\phi_1 P_{k_3}Q_{<k_3}\partial^\nu\phi_3]||_{N[0]}.$$

Note that the $(,)$ in the first summand on the right-hand side is at modulation $< 2^{k_3+O(1)}$. Now use theorem 3.1, as well as theorem 3.3 for the first summand; also, use theorem 4.1 for the 2nd summand. This concludes case **(B)**.

(C): High-Low interactions. $k_3 < -10$. This case is still simpler because ∇^{-1} falls on a high-frequency term. It is therefore left out. This finishes the proof of the theorem. ■

5. A quadrilinear estimate

Finally, we prove the following statement:

Theorem 5.1. *Let ϕ_i, $i = 1, 2, 3, 4$, be Schwartz functions on $\mathbf{R}^{3+1}$ satisfying $\max_i ||P_k\phi_i||_{S[k]} \leq Cc_k\ \forall k \in \mathbf{Z}$ for a sufficiently flat frequency envelope $\{c_k\}$. Then we have the inequality*

$$||P_0[\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_\nu\phi_1 R_i\phi_2 - R_i\phi_1 R_\nu\phi_2)R_j\phi_3)\partial^\nu\phi_4$$
$$-\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_j\phi_1 R_i\phi_2 - R_i\phi_1 R_j\phi_2)R_\nu\phi_3)\partial^\nu\phi_4]||_{N[0]}$$
$$\leq Cc_0.$$

Remark: The preceding trilinear null-form estimate implies a similar statement. Conversely, the immediately preceding inequality follows implicitly from a statement like the one of the trilinear null-form estimates. We omit this more precise form since it is obfuscating in this context.

Proof : We commence with **low-high interactions**, in the sense that $\partial^\nu\psi_4$ is at frequency ~ 1. Thus we consider expressions

$$P_0[\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)R_j P_{k_3}\phi_3)\partial^\nu P_{k_4}\phi_4$$
$$-\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_j P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_j P_{k_2}\phi_2)R_\nu P_{k_3}\phi_3)\partial^\nu P_{k_4}\phi_4],$$

where $k_4 \in [-10, 10]$. W. l. o. g. assume that $k_1 \geq k_2$. We may furthermore assume that $k_1 < -20$. For if not, we decompose

$$P_0[\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)R_j P_{k_3}\phi_3)\partial^\nu P_{k_4}\phi_4]$$
$$= P_0[\sum_{k_3<k-20<0} P_k N\partial^\nu P_{k_4}\phi_4 + \sum_{30>k+10>k_3\geq k-20} P_k N\partial^\nu P_{k_4}\phi_4$$
$$+\sum_{k_3\geq k+10} P_k N\partial^\nu P_{k_4}\phi_4],$$

where $N = \triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)R_j P_{k_3}\phi_3)$. Each of the three preceding summands is straightforward and can be estimated in $L_t^1\dot{H}^{-\frac{1}{2}}$. For example, we have

$$||P_0[\sum_{k_3<k-20<0} P_k N\partial^\nu P_{k_4}\phi_4||_{L_t^1 L_x^2}$$
$$\leq C\sum_{k_3<k-20<0} 2^{-k}||\triangle^{-1}\partial_i(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)||_{L_t^2 L_x^{4-}}$$
$$||R_j P_{k_3}\phi_3||_{L_t^4 L_x^\infty}||\partial^\nu P_{k_4}\phi_4||_{L_t^4 L_x^{4+}}$$
$$\leq C\sum_{k_3<k-20<0} 2^{-k}2^{k-k_1}2^{\frac{k_2-k_1}{2}}2^{-\frac{k}{4-}}2^{\frac{3}{4}k_3}\prod_i ||P_{k_i}\phi_i||_{S[k_i]}.$$

It is now possible to sum over the indicated ranges of k_i, k to obtain the desired inequality. Similarly, we may assume that $k_3 < -10$, say. *We next claim that we may assume that* $k_3 < k_1 + 10$. If the opposite is the case, we may represent our null-form schematically as

$$P_0[\nabla^{-1}(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)\nabla^{-1}P_{k_3}\phi_3\partial^\nu P_{k_4}\phi_4]$$
$$= P_0[\nabla^{-1}(R_j P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_j P_{k_2}\phi_2)\nabla^{-1}P_{k_3}R_\nu\phi_3\partial^\nu P_{k_4}\phi_4].$$

The 2nd term is straightforward to estimate on account of theorem 3.3, theorem 3.1. For the first term, we reduce $(P_{k_3}\nabla^{-1}\phi_3\partial^\nu P_{k_4}\phi_4)$ to modulation $< 2^{k_1+100}$ as follows: assume its modulation is of dyadic size $\sim 2^l$, $l > k_1 + 100$. Then either the output is at comparable modulation, or else $\nabla^{-1}(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - ...)$ is at least at comparable modulation. In the former case, we estimate

$$
\begin{aligned}
&||P_0 Q_{l+O(1)}[\nabla^{-1}(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2) \\
&\qquad\qquad\qquad\qquad Q_l(\nabla^{-1} P_{k_3}\phi_3 \partial^\nu P_{k_4}\phi_4)]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}} \\
&\leq C 2^{-\frac{l}{2}} 2^{k_1-\frac{k_3}{2}} 2^{\frac{k_2-k_1}{2}} \prod_i ||P_{k_i}\phi_i||_{S[k_i]}
\end{aligned}
$$

on account of theorem 3.1, and this can be summed over $l > k_1 + 100$ as well as k_i in the indicated ranges. In the latter case, we use the inequality

$$
\begin{aligned}
&||Q_{\geq l+O(1)}\nabla^{-1}(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)||_{L_t^2 L_x^\infty} \\
&\qquad\qquad\qquad\qquad \leq C 2^{\frac{k_1}{2}} 2^{\frac{k_1-l}{2}} 2^{\frac{k_2-k_1}{2}} \prod_{i=1,2} ||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

Having reduced $(\nabla^{-1} P_{k_3}\phi_3 \partial^\nu P_{k_4}\phi_4)$ to modulation $< 2^{k_1+100}$ and taking advantage of the fact that $k_3 < -10$, we may replace the term under consideration modulo an easily controlled error term by

$$
P_0[\sum_{i=1}^3 \triangle^{-1}\partial_i(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)\partial^\nu Q_{<k_1+100}(\nabla^{-1} P_{k_3}\phi_3 P_{k_4}\phi_4)].
$$

Then we can combine theorem 3.1 as well as theorem 4.2 to estimate:

$$
\begin{aligned}
&||P_0[\sum_{i=1}^3 \triangle^{-1}\partial_i(R_\nu P_{k_1}\phi_1 R_i P_{k_2}\phi_2 - R_i P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2) \\
&\qquad\qquad\qquad\qquad \partial^\nu Q_{<k_1+100}(\nabla^{-1} P_{k_3}\phi_3 P_{k_4}\phi_4)]||_{N[0]} \\
&\leq C 2^{\delta(k_2-k_1)} ||Q_{<k_1+100}(\nabla^{-1} P_{k_3}\phi_3 P_{k_4}\phi_4)||_{\dot{X}_0^{\frac{1}{2},\frac{1}{2},1}} \prod_{i=1,2} ||P_{k_i}\phi_i||_{S[k_i]} \\
&\leq C 2^{\delta_1(k_2-k_1)} 2^{\delta_2(k_1-k_3)} \prod_i ||P_{k_i}\phi_i||_{S[k_i]}.
\end{aligned}
$$

This shows that we only need to consider the case $k_1 < -20, k_3 < -10$ and $k_3 < k_1 + 10$. We further observe that we may assume that the output as well as the large frequency input $P_{k_4}\phi_4$ are at modulation $< 2^{k_1}$: this is straightforward and left for the reader. Now we invoke the following *null-form identity*:

$$\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_\nu\psi_1R_i\psi_2-R_i\psi_1R_\nu\psi_2)R_j\psi_3)\partial^\nu\psi_4$$
$$-\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_j\psi_1R_i\psi_2-R_i\psi_1R_j\psi_2)R_\nu\psi_3)\partial^\nu\psi_4$$
$$=\sum_{i,j=1}^{3}\Box[\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(\nabla^{-1}\psi_1R_i\psi_2)R_j\psi_3)\psi_4]$$
$$-\sum_{i,j=1}^{3}\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(\nabla^{-1}\psi_1R_i\psi_2)R_j\psi_3)\Box\psi_4$$
$$-\sum_{i,j=1}^{3}\Box[\triangle^{-1}\partial_j(\triangle^{-1}\partial_i(\nabla^{-1}\psi_1R_i\psi_2)R_j\psi_3)]\psi_4$$
$$-\sum_{j=1}^{3}\triangle^{-1}\partial_j(\nabla^{-1}\psi_1(R_\nu\psi_2R_j\psi_3-R_j\psi_2R_\nu\psi_3))\partial^\nu\psi_4$$
$$-\sum_{i=1}^{3}\triangle^{-1}\partial_i(\nabla^{-1}\psi_1R_i\psi_2)R_\nu\psi_3\partial^\nu\psi_4.$$

We substitute the appropriately microlocalized inputs. It is now entirely straightforward to estimate all terms on the right-hand side of the equality sign, using bilinear estimates proved earlier, except possibly the fourth term *in the case of high-high interactions within the outer* (,): More precisely, if we focus on the schematically written term[13]

$$P_0[\nabla^{-1}(\nabla^{-1}P_{k_1}\phi_1P_{k_3}\phi_3R_\nu P_{k_2}\phi_2)\partial^\nu P_{k_4}\phi_4],$$

then the only case not immediately covered by our earlier bilinear estimates is the following:

$$\sum_{k<k_2-10}P_0[\nabla^{-1}P_k(\nabla^{-1}P_{k_1}\phi_1P_{k_3}\phi_3R_\nu P_{k_2}\phi_2)\partial^\nu P_{k_4}\phi_4],$$

i. e. there is destructive resonance between $P_{k_2}R_\nu\phi_2$ and $(\nabla^{-1}P_{k_1}\phi_1P_{k_3}\phi_3)$. For this case we observe that we have the identity

$$P_k(R_\nu P_{k_2}\phi_2(\nabla^{-1}P_{k_1}\phi_1P_{k_3}\phi_3))$$
$$=\sum_{\omega_{1,2}\in K_{k-k_2-10},\,\mathrm{dist}(\omega_1,-\omega_2)\leq 2^{k-k_2+O(1)}}P_k(R_\nu P_{k_2,\omega_1}\phi_2P_{k_2+O(1),\omega_2}(\nabla^{-1}P_{k_1}\phi_1P_{k_3}\phi_3)).$$

Now we observe the following simple consequence of (5)[14]: $\forall\epsilon>0$ we have the

[13] We have omitted the localizers $Q_{<k_1}$ here since they are unnecessary for this term.
[14] Argue as in the proof of theorem 3.3.

inequality

$$||\partial^{\nu} P_{k_4}\phi_4 R_{\nu} P_{k_2,\omega_1}\phi_2||_{L_t^2 L_x^2} \leq C_{\epsilon} 2^{\frac{k_2}{2}} 2^{(1-\epsilon)(k-k_2)} \prod_{i=1,2} ||P_{k_i}\phi_i||_{S[k_i]}.$$

Also, we record the following consequences of the *improved Bernstein's inequality* as well as the *honest Bernstein's inequality*, respectively:

$$||P_{k_2+O(1),\omega_2} Q_j(\nabla^{-1} P_{k_1}\phi_1 P_{k_3}\phi_3)||_{L_t^2 L_x^\infty} \leq C 2^{\frac{j-k_2}{2+}} 2^{\frac{3k_2}{2}} 2^{\frac{k_3-3k_1}{2}} \prod_{i=1,3} ||P_{k_i}\phi_i||_{S[k_i]},$$

$$||P_{k_2+O(1),\omega_2} Q_j(\nabla^{-1} P_{k_1}\phi_1 P_{k_3}\phi_3)||_{L_t^2 L_x^\infty} \leq C 2^{k-k_2} 2^{\frac{3k_2}{2}} 2^{\frac{k_3-3k_1}{2}} \prod_{i=1,3} ||P_{k_i}\phi_i||_{S[k_i]},$$

whence interpolation yields $\forall \epsilon > 0$ and suitable $\mu(\epsilon) > 0$

$$\begin{aligned} &||P_{k_2+O(1),\omega_2} Q_j(\nabla^{-1} P_{k_1}\phi_1 P_{k_3}\phi_3)||_{L_t^2 L_x^\infty} \\ &\qquad \leq C_{\epsilon} 2^{\mu(\epsilon)(j-k_2)} 2^{(1-\epsilon)(k-k_2)} 2^{\frac{3k_2}{2}} 2^{\frac{k_3-3k_1}{2}} \prod_{i=1,3} ||P_{k_i}\phi_i||_{S[k_i]}. \end{aligned}$$

More precisely, using Plancherel's theorem and identical reasoning, we get

$$\begin{aligned} &\Big(\sum_{\omega \in K_{\frac{k-k_2}{2}-10}} ||P_{k_2+O(1),\omega} Q_j(\nabla^{-1} P_{k_1}\phi_1 P_{k_3}\phi_3)||_{L_t^2 L_x^\infty}^2\Big)^{\frac{1}{2}} \\ &\qquad \leq C_{\epsilon} 2^{\mu(\epsilon)(j-k_2)} 2^{(1-\epsilon)(k-k_2)} 2^{\frac{3k_2}{2}} 2^{\frac{k_3-3k_1}{2}} \prod_{i=1,3} ||P_{k_i}\phi_i||_{S[k_i]}. \end{aligned}$$

Now we use the fact that the multiplier $P_k \nabla^{-1}$ is given by convolution with a kernel $a(.)$ of L^1-mass $\sim 2^{-k}$. Thus we have the identity

$$\begin{aligned} &P_0[\nabla^{-1} P_k(\nabla^{-1} P_{k_1}\phi_1 P_{k_3}\phi_3 R_{\nu} P_{k_2}\phi_2)\partial^{\nu} P_{k_4}\phi_4] \\ &= \sum_{\omega_{1,2} \in K_{k-k_2-10},\, \mathrm{dist}(\omega_1,-\omega_2) \leq 2^{k-k_2+O(1)}} P_0[\nabla^{-1} P_k(P_{k_2+O(1),\omega_2}(\nabla^{-1} P_{k_1}\phi_1 P_{k_3}\phi_3) \\ &\qquad\qquad R_{\nu} P_{k_2,\omega_1}\phi_2)\partial^{\nu} P_{k_4}\phi_4] \\ &= \sum_{\omega_{1,2} \in K_{k-k_2-10},\, \mathrm{dist}(\omega_1,-\omega_2) \leq 2^{k-k_2+O(1)}} \int_{\mathbf{R}^3} a(y) T_y(P_{k_2+O(1),\omega_2}(\nabla^{-1} P_{k_1}\phi_1 P_{k_3}\phi_3) \\ &\qquad\qquad R_{\nu} P_{k_2,\omega_1}\phi_2)\partial^{\nu} P_{k_4}\phi_4] dy, \end{aligned}$$

where T_y refers to the translation operator $T_y f(x) := f(x-y)$. We further use the fact that T_y and microlocalization commute to infer from the preceding that

$$||P_0[\nabla^{-1}P_k(\nabla^{-1}P_{k_1}\phi_1 P_{k_3}\phi_3 R_\nu P_{k_2}\phi_2)\partial^\nu P_{k_4}\phi_4]||_{L_t^1 L_x^2}$$
$$\leq C \sum_{\omega_{1,2}\in K_{k-k_2-10},\, \mathrm{dist}(\omega_1,-\omega_2)\leq 2^{k-k_2+O(1)}}$$
$$\int_{\mathbf{R}^3} a(y)||P_{k_2+O(1),\omega_2}T_y(\nabla^{-1}P_{k_1}\phi_1 P_{k_3}\phi_3)||_{L_t^2 L_x^\infty}||R_\nu P_{k_2,\omega_1}T_y\phi_2\partial^\nu P_{k_4}\phi_4||_{L_t^2 L_x^2}\,dy.$$

Now use Cauchy-Schwartz and the definition of the $S[k]$, as well as the translation invariance of all Banach spaces used. We may also assume that $P_{k_2}R_\nu\phi_2$ and $\nabla^{-1}P_{k_1}\phi_1 P_{k_3}\phi_3$ are at modulation $< 2^{k_2+O(1)}$, since the opposite cases are straightforward. From the previous estimates, we get

$$||P_0[\nabla^{-1}P_k(\nabla^{-1}P_{k_1}\phi_1 P_{k_3}\phi_3 R_\nu P_{k_2}\phi_2)\partial^\nu P_{k_4}\phi_4]||_{L_t^1 L_x^2}$$
$$\leq C_\epsilon 2^{-k}2^{2(1-\epsilon)(k-k_2)}2^{2k_2}2^{\frac{k_3}{2}-\frac{3k_1}{2}}\prod_i ||P_{k_i}\phi_i||_{S[k_i]}.$$

It is now straightforward to sum over the appropriate ranges of k, k_i, $i = 1\ldots 4$ to obtain the desired result. This completes the **low-high case**. The remaining situations ($k_4 > 10$, $k_4 < -10$) are significantly simpler. For example, consider **high-high interactions** in the sense that $k_4 > 10$. The corresponding term can be morally[15] rewritten as

$$P_0[P_{k_4+O(1)}(\nabla^{-1}(R_\nu P_{k_1}\phi_1 R_j P_{k_2}\phi_2 - R_j P_{k_1}\phi_1 R_\nu P_{k_2}\phi_2)P_{k_3}\phi_3)P_{k_4}\phi_4].$$

We decompose this as

$$\sum_{k<k_4-C} P_0[P_{k_4+O(1)}(P_k Q_{\nu j}(P_{k_1}\phi_1, P_{k_2}\phi_2)P_{k_3}\phi_3)P_{k_4}\phi_4]$$
$$+ \sum_{k\in[k_4-C,k_4+C]} P_0[P_{k_4+O(1)}(P_k Q_{\nu j}(P_{k_1}\phi_1, P_{k_2}\phi_2)P_{k_3}\phi_3)P_{k_4}\phi_4]$$
$$+ \sum_{k>k_4-C} P_0[P_{k_4+O(1)}(P_k Q_{\nu j}(P_{k_1}\phi_1, P_{k_2}\phi_2)P_{k_3}\phi_3)P_{k_4}\phi_4].$$

For example, the first sum is estimated by getting rid of the disposable multiplier $P_{k_4+O(1)}$ (replacing $P_{k_3}\phi_3$ by translates) and using theorem 3.1, theorem 3.2; one needs to further distinguish between $k <> 0$. We leave the simple details for the reader. The other terms are estimated similarly. ∎

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