

**MATHEMATICS 350 FALL 2008 (SHATZ)**  
**Assignment I, September 8, 2008. Due September 22, 2008**

HAEC FABULA DOCET  
by Robert Frost

A Blindman by the name of La Fontaine,  
Relying on himself and on his cane,  
Came tap-tap-tapping down the village street,  
The apogee of human blind conceit.  
Now just ahead of him was seen to yawn  
A trench where water pipes were laying on.  
The Blindman might have found it with his ferrule,  
But someone overanxious at his peril  
Not only warned him with a loud command  
But ran against him with a staying hand.  
Enraged at what he could but think officious,  
The Blindman missed him with a blow so vicious  
He gave his own poor iliac a wrench  
And plunged himself head foremost in the trench:  
Where, with a glee no less for being grim,  
The workmen all turned to and buried him.

MORAL

The moral is it hardly need be shown,  
All those who try to go it sole alone,  
Too proud to be beholden for relief,  
Are absolutely sure to come to grief.

**You should be choosing your groups for the B HW. By Friday 9/12/08, I want (in writing) a list of groups, each group to list its members in alphabetical order.**

**A PROBLEMS (NOT TO BE HANDED IN).**

AI Show that for each positive integer,  $n$ , the number  $2^{n+2} + 3^{2n+1}$  is exactly divisible by 7. Similarly, show that for each *odd* positive integer,  $n$ , the number  $2^{n+3} + 8^{2n+1}$  is exactly divisible by 11. Can you generalize these statements?

AII Consider the sum

$$1 + 2 + \cdots + n = \sum_{j=1}^n j.$$

Prove, by induction, a formula evaluating this sum. Deduce from your formula a formula for the sum  $1 + 3 + 5 + 7 + \cdots + 2n + 1 = \sum_{j=0}^n (2j + 1)$ . Similarly, find and prove formulae for the sums:  $1^k + 2^k + \cdots + n^k$ , when  $k$  is 2 or 3; as well as formulae for the sums:  $1^k + 3^k + 5^k + \cdots + (2n + 1)^k$ , when  $k$  is 2 or 3. Can you guess the shape of the formulae for general  $k$ ? Try to give a proof (just prove that the shape of the formula is correct in each case).

AIII In how many different ways can you change a dollar? That is, in how many different ways can you pay 100 cents using five different kinds of coins: pennies, nickels, dimes, quarters and half-dollars? Someone owns a set of eight weights: 1, 1, 2, 5, 10, 20, 50 grams. In how many ways can 78 grams be composed of such weights? (Replacing one weight by another of the same value counts as a different way.)

AIV) Suppose you own a balance and a set of weights of: 1, 2, 4, 8, 16,  $\dots$ ,  $2^n$ ,  $\dots$  grams. Show that given any unknown weight of  $r$  grams, where  $r$  is a positive integer, there is exactly one combination of your weights which will balance the unknown weight and so determine  $r$ . Give an algorithm to find this combination.

### B PROBLEMS (TO BE HANDED IN-GROUP WORK).

BI Given a finite sequence

$$a_1 \leq a_2 \leq \dots \leq a_n$$

of real numbers, we say it has a *perfect median* if and only if

$$(\exists p)(1 \leq p \leq n)(a_1 + a_2 + \dots + a_{p-1} = a_{p+1} + a_{p+2} + \dots + a_n).$$

For example, in the finite sequence,

$$1 \leq 2 \leq \dots \leq 8$$

the number 6 is the perfect median because

$$1 + 2 + 3 + 4 + 5 = 15 = 7 + 8.$$

a) For which positive integers,  $n$ , does the finite sequence

$$1 \leq 2 \leq 3 \leq \dots \leq n$$

possess a perfect median? Your analysis should be powerful enough to answer the following questions:

1) Are there infinitely many  $n$  for which our sequence has a perfect median?

2) Can you construct by an algorithm all such possible  $n$ ?

(You may wish to begin with an elementary analysis which allows you to find the smallest  $n$  beyond  $n = 8$  so that

$$1 \leq 2 \leq \dots \leq n$$

has a perfect median. Failing this, you may wish to compute such an  $n$  by hand or by computer program.)

b) For our sequence of integers, without giving a formula to find a perfect median (assuming such exists),  $p$ , show there is exactly one such median; we will label it  $p(n)$ .

c) Write  $S$  for the subset of the positive integers consisting of those  $n$  having a perfect median. Show there is an explicit constant,  $K$ , which you are exactly and explicitly to determine, such that

$$Kp(n) \sim n, \quad n \in S \quad \text{and} \quad n \rightarrow \infty.$$

Here,  $Kp(n) \sim n$  means

$$\lim_{n \rightarrow \infty} Kp(n)/n = 1.$$

BII Recall that if  $a, b$  are real numbers, the interval  $(a, b)$  consists of all real numbers  $x$  so that  $a < x < b$ . We call a real valued function,  $f$ , defined on  $(a, b)$  *convex* provided it satisfies

$$(\forall x, y \in (a, b))(\forall \lambda \in (0, 1))(f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)).$$

Suppose now that  $f$  is convex on  $(a, b)$  and  $t_1, \dots, t_r$  are arbitrary points of  $(a, b)$ . Given arbitrary positive real numbers, say  $p_1, \dots, p_r$ , prove that

$$f\left(\frac{p_1 t_1 + \dots + p_r t_r}{p_1 + \dots + p_r}\right) \leq \frac{p_1 f(t_1) + \dots + p_r f(t_r)}{p_1 + \dots + p_r}.$$

BIII Galileo found that the positive integers possessed a proper subset (the even positive integers) so that there is a bijection between the “small” subset and the big parent set of positive integers. He regarded this as a paradox, but, actually, it is a characteristic property of infinite sets—we’ve none of us ever “seen” an infinite set. So, for this problem, we call a set *finite* whenever it is impossible to find a proper subset of itself which is in bijection with the whole set. And, we’ll call a set *infinite* whenever it is NOT finite. Prove the following statements (based on these definitions):

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- a) If a set,  $S$ , possesses a subset,  $T$ , and  $T$  is infinite, then  $S$  is also infinite.
- b) If  $X$  and  $Y$  are both finite then so is their union.
- c) Given a non-empty set,  $X$ , one and only one of the following is true:  
There is an injection from the whole set of positive integers to  $X$ .  
There is a unique positive integer,  $n$ , so that the set  $1, 2, \dots, n$  injects into  $X$ , but the set  $1, 2, \dots, n + 1$  possesses no injection into  $X$ .

Which of these statements corresponds to the statement:  $X$  is finite and which to the statement:  $X$  is infinite?