MATHEMATICS 350 FALL 2008 (SHATZ)
Assignment III, October 6, 2008. Due October 20, 2008

Some aphorisms from antiquity and the renaissance:

Itidem homines exercendo videmus conteri. Inertia atque torpedo plus detrementi facit, quam exercitio.
(Cato the Elder; Carmen de Moribus)

Non omnia possumus omnes. (Virgil, 8th Eclogue)

Le coeur a ses raisons que la raison ne connait point.
L'homme n'est qu'un roseau, le plus faible de la nature; mais c'est un roseau pensant.
(Pascal; Pensées)

Le pénible fardeau de n'avoir rien à faire.
(Boileau; Epîtres à M. Racine, # 9.)

A PROBLEMS (NOT TO BE HANDED IN).

AI Show that $1 + \frac{1}{2} + \cdots + \frac{1}{n} \notin \mathbf{Z}$ if $n > 1$ and that

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \notin \mathbf{Q}.$$

AII Suppose that A is a given integral domain–of which we have many examples. We will call A a *Euclidean Domain* provided it comes equipped with a function N satisfying:

1) N is defined on all elements of A except 0 and $N(a)$ is a non-negative integer,

2) If $a|b$, then $N(a) \le N(b)$,

3) For all a and b with $a \neq 0$, we can find q, r so that $b = aq + r$ and either $r = 0$ or $N(r) < N(a)$.

(Notice that $\mathbf{Z}$ is a Euclidean Domain, with $N(a) = |a|$.) By imitating the proofs of class, show that every non-zero, non-unit element of A is a product of finitely many irreducible elements, that greatest common divisors exist, that a g.c.d. of two elements is a linear combination of those elements and, finally, that a Euclidean Domain is a UFD.

AIII For the real numbers, $\mathbf{R}$, one of the properties we find is that each positive real number is less than some integer (which integer will depend on the positive real number chosen). The first recorded person to call attention to this property of reals *vs.* integers was the Greek mathematician Archimedes (fl. third century BC)–he did this in connection with his discovery of the principle of the lever. In fact, it is reported that he told his royal patron: "Give me a place to stand and I will move the earth." Assume Archimedes' property as given above and prove: Between any two distinct real numbers there is a rational number.

AIV Assume the ABC conjecture (cf. Problem BIII, Assignment II), and on the strength of this conjecture answer the following question: Let p and q be prime numbers with $p < q$, and consider integers $c(m,n) = p^m + q^n$ as m and n vary. Can there be infinitely many numbers of the form $2^r\pi^s$ (with π a prime) among the $c(m,n)$? (If $p = 2$, skip the factor 2^r in $c(m,n)$.)

B PROBLEMS (TO BE HANDED IN–GROUP WORK).

BI Let us write $\mathbf{Z}[\mathrm{i}]$ for the set of all complex numbers of the form $a + ib$ where a, b are integers; here i is the complex number whose square is -1. Further, write $\mathbf{Z}[\sqrt{2}]$ for the set of all real numbers of the form $a + b\sqrt{2}$ where a, b are integers. You easily see that both $\mathbf{Z}[i]$ and $\mathbf{Z}[\sqrt{2}]$ are integral domains with their ordinary laws of addition and multiplication. Find all the units in these two domains. Show that both of these domains are Euclidean Domains. (The real difficulty here is in your choice of the function N in each case and in the proof that the remainder, r, satisfies $N(r) < N(a)$ if $r \neq 0$. For this, I suggest you plot the points of these domains on graph paper and use your geometric intuition to guide you to the correct r.)

BII a) A complex number, α, is an *algebraic number of degree* n provided there is a polynomial of degree n with integer coefficients, say

$$P(X) = a_0X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n,$$

so that $P(\alpha) = 0$ and n is the smallest degree of such a polynomial that has α as a root. (For example, $\sqrt{5}$ is an algebraic number of degree 2.) Now, by Problem AIII above, we may always approximate α by a rational number $\frac{r}{s}$; show, however, that we always have

$$|\alpha - \frac{r}{s}| > \frac{K(\alpha)}{s^n},$$

where $K(\alpha)$ is a positive constant independent of r, s.

b) A complex number is called *transcendental* if it is not an algebraic number of ANY degree. Choose any sequence of positive integers, say $a_1, a_2, \cdots, a_r \cdots$, and form the number β given by

$$\beta = \sum_{i=1}^{\infty} \frac{(-1)^{a_i}}{10^{i!}}.$$

Show that β is always a transcendental number.

BIII a) A commutative ring is called a *field* provided each non-zero element is a unit (that is, each non-zero element has an inverse under multiplication). Examples are: $\mathbf{Q}, \mathbf{R}, \mathbf{C}$ as well as $\mathbf{F}_2$ and $\mathbf{F}_3$. Let k stand for a field and write $k[X]$ for the integral domain of all polynomials in the variable X with coefficients in the field k. Show that $k[X]$ is a Euclidean Domain; hence all the results of class are valid for it. In particular, irreducible polynomials are prime elements and we have unique factorization–that is, $k[X]$ is a UFD.

b) If A is a ring, denote by $A[X]$ the collection of all polynomials in the variable X with coefficients from A. We know this a ring, show it is a domain provided A is itself a domain. Now we have the two domains: $\mathbf{Z}[X]$ and $k[X, Y]$, where k is a field. It turns out both of these are UFD's. Prove, however, neither of these is a Euclidean Domain.