Guide to Math 241

Part I. Solving PDEs in bounded regions

Step 1. Apply separation of variables to get ordinary differential equation(ODE)s for each variable.
Step 2. Interprete boundary condition(BC)s as restrictions for the corresponding ODEs, then solve each ODE that has 2 restrictions as an eigenvalue problem.
Step 3. Plug in the eigenvalues you get from step 2 and solve the rest of the ODEs, so you can form the general solution.
Step 4. Determine the coefficients in the general solution by initial condition(IC)s.

In step 1, we have the following cases:

(1) One-Dimensional interval: $0 \leq x \leq L$

For heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, let $u(x,t) = X(x)T(t)$, the ODEs are:

$$T'(t) = -k\lambda T(t)$$

$$X''(x) = -\lambda X(x)$$

For wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, let $u(x,t) = X(x)T(t)$, the ODEs are:

$$T''(t) = -c^2\lambda T(t)$$

$$X''(x) = -\lambda X(x)$$

(2) Two-Dimensional rectangle: $0 \leq x \leq L$**,** $0 \leq y \leq H$

For heat equation $\frac{\partial u}{\partial t} = k\Delta u$, let $u(x,y,t) = X(x)Y(y)T(t)$, the ODEs are:

$$T'(t) = -k\lambda T(t)$$

$$X''(x) = -\mu X(x)$$

$$Y''(y) = -(\lambda - \mu)Y(y)$$

For wave equation $\frac{\partial^2 u}{\partial^2 t} = c^2\Delta u$, let $u(x,y,t) = X(x)Y(y)T(t)$, the ODEs are:

$$T''(t) = -c^2\lambda T(t)$$

$$X''(x) = -\mu X(x)$$

$$Y''(y) = -(\lambda - \mu)Y(y)$$

For Laplace's equation $\Delta u = 0$, let $u(x,y) = X(x)Y(y)$, the ODEs are:

$$X''(x) = -\lambda X(x)$$

$$Y''(y) = \lambda Y(y)$$

(3) Two-Dimensional disk(annulus) or a sector of a disk(annulus), where the spacial part is described by polar coordinates (r, θ)

For heat equation $\frac{\partial u}{\partial t} = k\Delta u$, let $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, the ODEs are:

$$T'(t) = -k\lambda T(t)$$

$$\Theta''(\theta) = -\mu\Theta(\theta)$$

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - \mu)R(r) = 0$$

For wave equation $\frac{\partial^2 u}{\partial^2 t} = c^2\Delta u$, let $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, the ODEs are:

$$T''(t) = -c^2\lambda T(t)$$

$$\Theta''(\theta) = -\mu\Theta(\theta)$$

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - \mu)R(r) = 0$$

For Laplace's equation $\Delta u = 0$, let $u(r, \theta) = R(r)\Theta(\theta)$, the ODEs are:

$$\Theta''(\theta) = -\lambda\Theta(\theta)$$

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0$$

(4) Two-Dimensional sphere or a sector of a sphere, where the spacial part is described by spherical coordinates (ϕ, θ)

For heat equation $\frac{\partial u}{\partial t} = k\Delta u$, let $u(\phi, \theta, t) = \Phi(\phi)\Theta(\theta)T(t)$, the ODEs are:

$$T'(t) = -k\lambda T(t)$$

$$\Theta''(\theta) = -\mu\Theta(\theta)$$

$$\big(\sin\phi \ \Phi'(\phi)\big)' + \Big(\lambda \sin\phi - \frac{\mu}{\sin\phi}\Big)\Phi(\phi) = 0$$

For wave equation $\frac{\partial^2 u}{\partial^2 t} = c^2\Delta u$, let $u(\phi, \theta, t) = \Phi(\phi)\Theta(\theta)T(t)$, the ODEs are:

$$T''(t) = -c^2\lambda T(t)$$

$$\Theta''(\theta) = -\mu\Theta(\theta)$$

$$\big(\sin\phi \ \Phi'(\phi)\big)' + \Big(\lambda \sin\phi - \frac{\mu}{\sin\phi}\Big)\Phi(\phi) = 0$$

For Laplace's equation $\Delta u = 0$, let $u(\phi, \theta) = \Phi(\phi)\Theta(\theta)$, the ODEs are:

$$\Theta''(\theta) = -\lambda\Theta(\theta)$$

$$\big(\sin\phi \ \Phi'(\phi)\big)' - \frac{\lambda}{\sin\phi}\Phi(\phi) = 0$$

(5) Three-Dimensional cylinder(cylindrical shell) or a sector of a cylinder (cylindrical shell), where the spacial part is described by cylindrical coordinates (r, θ, z)

For heat equation $\frac{\partial u}{\partial t} = k\Delta u$, let $u(r, \theta, z, t) = R(r)\Theta(\theta)Z(z)T(t)$, the ODEs are:

$$T'(t) = -k\lambda T(t)$$

$$\Theta''(\theta) = -\mu\Theta(\theta)$$
$$Z''(z) = -\nu Z(z)$$
$$r^2 R''(r) + rR'(r) + [(\lambda - \nu)r^2 - \mu]R(r) = 0$$

For wave equation $\frac{\partial^2 u}{\partial^2 t} = c^2\Delta u$, let $u(r, \theta, z, t) = R(r)\Theta(\theta)Z(z)T(t)$, the ODEs are:

$$T''(t) = -c^2\lambda T(t)$$

$$\Theta''(\theta) = -\mu\Theta(\theta)$$
$$Z''(z) = -\nu Z(z)$$
$$r^2 R''(r) + rR'(r) + [(\lambda - \nu)r^2 - \mu]R(r) = 0$$

For Laplace's equation $\Delta u = 0$, let $u(r, \theta, z) = R(r)\Theta(\theta)Z(z)$, the ODEs are:

$$Z''(z) = \lambda Z(z)$$

$$\Theta''(\theta) = -\mu\Theta(\theta)$$
$$r^2 R''(r) + rR'(r) + (\lambda r^2 - \mu)R(r) = 0$$

(6) Three-Dimensional ball(spherical shell) or a sector of a ball(spherical shell), where the spacial part is described by spherical coordinates (r, ϕ, θ)

For heat equation $\frac{\partial u}{\partial t} = k\Delta u$, let $u(r, \phi, \theta, t) = R(r)\Phi(\phi)\Theta(\theta)T(t)$, the ODEs are:

$$T'(t) = -k\lambda T(t)$$

$$\Theta''(\theta) = -\mu\Theta(\theta)$$
$$\left(r^2 R'(r)\right)' + (\lambda r^2 - \nu)R(r) = 0$$
$$\left(\sin\phi\ \Phi'(\phi)\right)' + \left(\nu\sin\phi - \frac{\mu}{\sin\phi}\right)\Phi(\phi) = 0$$

For wave equation $\frac{\partial^2 u}{\partial^2 t} = c^2\Delta u$, let $u(r, \phi, \theta, t) = R(r)\Phi(\phi)\Theta(\theta)T(t)$, the ODEs are:

$$T''(t) = -c^2\lambda T(t)$$

$$\Theta''(\theta) = -\mu\Theta(\theta)$$
$$\left(r^2 R'(r)\right)' + (\lambda r^2 - \nu)R(r) = 0$$
$$\left(\sin\phi\ \Phi'(\phi)\right)' + \left(\nu\sin\phi - \frac{\mu}{\sin\phi}\right)\Phi(\phi) = 0$$

For Laplace's equation $\Delta u = 0$, let $u(r,\phi,\theta) = R(r)\Phi(\phi)\Theta(\theta)$, the ODEs are:

$$\Theta''(\theta) = -\lambda\Theta(\theta)$$

$$\big(r^2 R'(r)\big)' - \mu R(r) = 0$$

$$\big(\sin\phi\ \Phi'(\phi)\big)' + \Big(\mu\sin\phi - \frac{\lambda}{\sin\phi}\Big)\Phi(\phi) = 0$$

In step 2, we have the following cases:

(1) For ODE $X''(x) = -\lambda X(x)$

If $X(0) = 0$, $X(L) = 0$, then

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \cdots$$

If $X'(0) = 0$, $X'(L) = 0$, then

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, \cdots$$

If $X(-L) = X(L)$, $X'(-L) = X'(L)$, then

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}, \quad n = 0, 1, \cdots$$

If $X(0) = 0$, $X'(L) = 0$, then

$$\lambda_n = \frac{(n-\frac{1}{2})^2\pi^2}{L^2}, \quad X_n(x) = \sin\frac{(n-\frac{1}{2})\pi x}{L}, \quad n = 1, 2, \cdots$$

If $X'(0) = 0$, $X(L) = 0$, then

$$\lambda_n = \frac{(n-\frac{1}{2})^2\pi^2}{L^2}, \quad X_n(x) = \cos\frac{(n-\frac{1}{2})\pi x}{L}, \quad n = 1, 2, \cdots$$

(2) For ODE $r^2R''(r) + rR'(r) = -\lambda R(r)$

Setting $r = \mathrm{e}^t$ will change this equation to $R''(t) = -\lambda R(t)$, therefore we can relate it to the previous case.

If $R(a) = 0$, $R(b) = 0$, then

$$\lambda_n = \left(\frac{n\pi}{\ln\frac{b}{a}}\right)^2, \quad R_n(r) = \sin\frac{n\pi\ln\frac{x}{a}}{\ln\frac{b}{a}}, \quad n = 1, 2, \cdots$$

If $R'(a) = 0$, $R'(b) = 0$, then

$$\lambda_n = \left(\frac{n\pi}{\ln\frac{b}{a}}\right)^2, \quad R_n(r) = \cos\frac{n\pi\ln\frac{x}{a}}{\ln\frac{b}{a}}, \quad n = 0, 1, 2, \cdots$$

If $R(a) = R(b)$, $R'(a) = R'(b)$, then

$$\lambda_n = \left(\frac{2n\pi}{\ln\frac{b}{a}}\right)^2, \quad R_n(r) = A_n \sin\frac{2n\pi \ln\frac{x}{\sqrt{ab}}}{\ln\frac{b}{a}} + B_n \cos\frac{2n\pi \ln\frac{x}{\sqrt{ab}}}{\ln\frac{b}{a}}, \quad n = 0, 1, 2, \cdots$$

If $R(a) = 0$, $R'(b) = 0$, then

$$\lambda_n = \left(\frac{(n-\frac{1}{2})\pi}{\ln\frac{b}{a}}\right)^2, \quad R_n(r) = \sin\frac{(n-\frac{1}{2})\pi \ln\frac{x}{a}}{\ln\frac{b}{a}}, \quad n = 1, 2, \cdots$$

If $R'(a) = 0$, $R(b) = 0$, then

$$\lambda_n = \left(\frac{(n-\frac{1}{2})\pi}{\ln\frac{b}{a}}\right)^2, \quad R_n(r) = \cos\frac{(n-\frac{1}{2})\pi \ln\frac{x}{a}}{\ln\frac{b}{a}}, \quad n = 1, 2, \cdots$$

(3) For ODE $r^2 R''(r) + rR'(r) + (\lambda r^2 - \nu^2)R(r) = 0$

If $|R(0)| < +\infty$, $R(a) = 0$, then

$$\lambda_n = \left(\frac{z_{\nu,n}}{a}\right)^2, \quad R_n(r) = J_\nu(\sqrt{\lambda_n} r), \quad n = 1, 2, \cdots$$

where $J_\nu(z)$ is *Bessel function of the first kind*, and $z_{\nu,n}$ is the n-th zero of $J_\nu(z)$, ν can be any non-negative real number, not necessary an integer.

(4) For ODE $\big(\sin\phi\ \Phi'(\phi)\big)' + \big(\mu\sin\phi - \frac{m^2}{\sin\phi}\big)\Phi(\phi) = 0$

If $|\Phi(0)| < +\infty$, $|\Phi(\pi)| < +\infty$, then

$$\mu_n = n(n+1), \quad \Phi_n(\phi) = P_n^m(\cos\phi), \quad n \geq m$$

where $P_n^m(x)$ is *associated legendre function(sperical harmonic) of first kind*, m and n are non-negative integers.

(5) For ODE $\big(r^2 R'(r)\big)' + \big(\lambda r^2 - n(n+1)\big)R(r) = 0$

If $|R(0)| < +\infty$, $R(a) = 0$, then

$$\lambda_n = \left(\frac{z_{n+\frac{1}{2},k}}{a}\right)^2, \quad R_n(r) = r^{-\frac{1}{2}} J_{n+\frac{1}{2}}\big(\sqrt{\lambda_n} r\big), \quad k = 1, 2, \cdots$$

where $z_{n+\frac{1}{2},k}$ is the k-th zero of $J_{n+\frac{1}{2}}(z)$.

(6) For regular Sturm-Liouville problem:

$$\big(p(x)\phi'(x)\big)' + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0, \quad a \leq x \leq b$$

$p(x)$, $q(x)$ and $\sigma(x)$ are continuous for $a \leq x \leq b$ and $p(x) > 0$, $\sigma(x) > 0$.

If $\beta_1\phi(a) + \beta_2\phi'(a) = 0$, $\beta_1\phi(a) + \beta_2\phi'(a) = 0$, then we can list eigenvalues in an increasing order $\lambda_1 < \lambda_2 < \cdots$, for each eigenvalue λ_n, we have a corresponding eigenfunction $\phi_n(x)$, and they are related by Rayleigh quotient.

In step 3, we have the following cases:

(1) For ODE $T'(t) = -\lambda T(t)$, the solution is $T(t) = c_1 \mathrm{e}^{-\lambda t}$

(2) For ODE $X''(x) = -\lambda X(x)$

If $\lambda > 0$, then

$$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

If $\lambda = 0$, then

$$X(x) = c_1 + c_2 x$$

If $\lambda < 0$, then

$$X(x) = c_1 \mathrm{e}^{\sqrt{-\lambda}x} + c_1 \mathrm{e}^{-\sqrt{-\lambda}x}$$

or we can write it as

$$X(x) = c_1 \cosh\sqrt{\lambda}x + c_2 \sinh\sqrt{\lambda}x$$

In all the cases above, we can do a shift for x, i.e., replace x by $x - a$ for a suibable a so as to fit the boundary restrictions.

(3) For ODE $r^2 R''(r) + rR'(r) = -\lambda R(r)$

If $\lambda > 0$, then

$$R(r) = c_1 \cos\left(\sqrt{\lambda}\ln r\right) + c_2 \sin\left(\sqrt{\lambda}\ln r\right)$$

If $\lambda = 0$, then

$$R(r) = c_1 + c_2 \ln r$$

If $\lambda < 0$, then

$$X(x) = c_1 r^{\sqrt{-\lambda}} + c_1 r^{-\sqrt{-\lambda}}$$

In all the cases above, we can do a scale for r, i.e., replace r by $\frac{r}{a}$ for a suibable a so as to fit the boundary restrictions.

(4) For ODE $r^2 R''(r) + rR'(r) + (\lambda r^2 - \nu^2)R(r) = 0, \ \ \nu \geq 0$

If $\lambda > 0$, then

$$R(r) = c_1 J_\nu(\sqrt{\lambda}r) + c_2 Y_\nu(\sqrt{\lambda}r)$$

where $Y_\nu(z)$ is *Bessel function of the second kind.*

If $\lambda = 0$, the equation becomes the one in case (2).

If $\lambda < 0$, then

$$R(r) = c_1 I_\nu(\sqrt{-\lambda}r) + c_2 K_\nu(\sqrt{-\lambda}r)$$

where $I_\nu(z)$ is *modified Bessel function of the first kind*, $K_\nu(z)$ is *modified Bessel function of the second kind.*

(5) For ODE $\left(r^2R'(r)\right)' + \left(\lambda r^2 - n(n+1)\right)R(r) = 0$

If $\lambda > 0$, then

$$R(r) = c_1 r^{-\frac{1}{2}} J_\nu(\sqrt{\lambda} r) + c_2 r^{-\frac{1}{2}} Y_\nu(\sqrt{\lambda} r)$$

If $\lambda = 0$, then

$$R(r) = c_1 r^n + c_2 r^{-n-1}$$

If $\lambda < 0$, then

$$R(r) = c_1 r^{-\frac{1}{2}} I_\nu(\sqrt{-\lambda} r) + c_2 r^{-\frac{1}{2}} K_\nu(\sqrt{-\lambda} r)$$

In step 4, we have the following cases:

(1) If $\phi_n(x)$ are orthogonal functions with weight $\sigma(x)$, i.e.

$$\int_a^b \phi_m(x)\phi_n(x)\sigma(x)\mathrm{d}x = 0, \quad m \neq n$$

and we expand f as

$$f(x) = \sum A_n \phi_n(x)$$

then

$$A_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\mathrm{d}x}{\int_a^b \phi_n^2(x)\sigma(x)\mathrm{d}x}$$

In particular, for trigonometric functions, the weight function $\sigma(x) = 1$; for Bessel functions, the weight function $\sigma(x) = x$.

(2) If $\phi_{m,n}$ are orthogonal functions with 2 variables ((x, y) or (r, θ)), and we expand f as

$$f = \sum\sum A_{m,n}\phi_{m,n}$$

then

$$A_{m,n} = \frac{\iint f\phi_{m,n}\ \mathrm{d}A}{\iint \phi_{m,n}^2\ \mathrm{d}A}$$

where $\mathrm{d}A = \mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$.

(3) If $\phi_{m,n,k}$ are orthogonal functions with 3 variables ((x, y, z) or (r, θ, z) or (r, ϕ, θ)), and we expand f as

$$f = \sum\sum\sum A_{m,n,k}\phi_{m,n,k}$$

then

$$A_{m,n,k} = \frac{\iiint f\phi_{m,n,k}\ \mathrm{d}V}{\iiint \phi_{m,n,k}^2\ \mathrm{d}V}$$

where $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z = r^2\sin\phi\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta$.

Part II. Solving PDEs in unbounded regions

Step 1. Apply Fourier transform (usually on x), to get an ODE.
Step 2. Solve this ODE.
Step 3. Take inverse Fourier transform of the solution in step 2, and express the result as convolution of initial condition and Fourier inverse of a function.
Step 4. Compute the inverse Fourier transform of a certain function and write the solution as an integration.

Note: If the domain of x is semi-infinite: $0 \leq x < \infty$, we first take odd/even extensions of ICs according to BC(if BC is of first type, e.g., $u(0,t) = 0$, then take odd extension; if BC is of second type, e.g., $\frac{\partial u}{\partial x}(0,t) = 0$, then take even extension), then use the 4 steps above to solve the PDE on $-\infty < x < \infty$ with extended ICs.

Definitions:

Denote $\mathcal{F}$ as Fourier transform of x variables, i.e. $\mathcal{F}$ takes a function $f(x)$ about x to a function $F(\omega)$ about ω, by

$$F(\omega) = \mathcal{F}[f](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{i\omega x}\mathrm{d}x$$

Its inverse is denoted as $\mathcal{F}^{-1}$, which is defined by

$$f(x) = \mathcal{F}^{-1}[F](x) = \int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{-i\omega x}\mathrm{d}\omega$$

Properties:

(1). Fourier transform and its inverse are linear:

$$\mathcal{F}[c_1 f + c_2 g] = c_1\mathcal{F}[f] + c_2\mathcal{F}[g], \quad \mathcal{F}^{-1}[c_1 F + c_2 G] = c_1\mathcal{F}^{-1}[F] + c_2\mathcal{F}^{-1}[G]$$

(2). Fourier transform and inverse Fourier transform are inverse to each other:

$$\mathcal{F}^{-1}[\mathcal{F}[f]] = f$$

(3). Fourier transform takes differential of x to multiplication by $-i\omega$:

$$\mathcal{F}\Big[\frac{\partial f}{\partial x}\Big] = -i\omega\mathcal{F}[f], \quad \mathcal{F}\Big[\frac{\partial^2 f}{\partial x^2}\Big] = -\omega^2\mathcal{F}[f]$$

(4). Fourier transform commutes with partial derivatives other than x:

$$\mathcal{F}\Big[\frac{\partial f}{\partial t}\Big] = \frac{\partial}{\partial t}\mathcal{F}[f], \quad \mathcal{F}\Big[\frac{\partial f}{\partial y}\Big] = \frac{\partial}{\partial y}\mathcal{F}[f]$$

(5). Inverse Fourier transform takes shift by α to multiplication by $\mathrm{e}^{i\alpha x}$:

$$\mathcal{F}^{-1}[F(\omega + \alpha)] = \mathrm{e}^{i\alpha x}\mathcal{F}^{-1}[F]$$

(6). Inverse Fourier transform takes multiplication to convolution:

$$\mathcal{F}^{-1}[FG] = \frac{1}{2\pi}\mathcal{F}^{-1}[F] * \mathcal{F}^{-1}[G]$$

where convolution ($*$) takes two functions f, g of x to a new function $f * g$ of x, it is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(\bar{x})g(x - \bar{x})\mathrm{d}\bar{x}$$

Useful indentities:

(1)

$$\mathcal{F}^{-1}[\mathrm{e}^{-\beta\omega^2}] = \sqrt{\frac{\pi}{\beta}}\mathrm{e}^{-\frac{x^2}{4\beta}}$$

(2)

$$\mathcal{F}^{-1}[\mathrm{e}^{i\alpha\omega}] = 2\pi\delta_\alpha(x)$$

where $\delta_\alpha(x)$ is the delta function, it is charactered by the property that when integrating with a function, it evaluates the function at α, i.e. $\int_{-\infty}^{\infty} \delta_\alpha(x)f(x)\mathrm{d}x = f(\alpha)$

(3)

$$\mathcal{F}^{-1}[\cos\alpha\omega] = \pi\big(\delta_\alpha(x) + \delta_{-\alpha}(x)\big)$$

(4)

$$\mathcal{F}^{-1}\Big[\frac{\sin\alpha\omega}{\omega}\Big] = \begin{cases} \pi & |x| < \alpha \\ 0 & |x| > \alpha \end{cases}$$

(5)

$$\mathcal{F}^{-1}[\mathrm{e}^{-\alpha|\omega|}] = \frac{2\alpha}{x^2 + \alpha^2}$$

(6)

$$f * \delta_\alpha = f(x - \alpha)$$

(7)

$$f * \mathrm{Rect}_\alpha = \int_{x-\alpha}^{x+\alpha} f(\bar{x})\mathrm{d}\bar{x}$$

where $\mathrm{Rect}_\alpha(x) = \begin{cases} 1 & |x| < \alpha \\ 0 & |x| > \alpha \end{cases}$

Part III. Finite difference numerical methods

For heat or wave equation on $0 \le x \le L$, we adopt the following notations:

$$
\begin{aligned}
&N : \text{a sufficient large integer}\\
&\Delta x = \frac{L}{N}\\
&x_j = j\Delta x, \quad j = 0, 1, \cdots, N\\
&\Delta t : \text{a chosen small increment of time}\\
&t_m = m\Delta t, \quad m = 0, 1, \cdots\\
&u_j^{(m)} = u(x_j, t_m)
\end{aligned}
$$

For a given PDE, we get a difference equation by:

$$
\begin{aligned}
\text{replacing } \frac{\partial^2 u}{\partial t^2} &\text{ by } \frac{u_j^{(m+1)} - 2u_j^{(m)} + u_j^{(m-1)}}{(\Delta t)^2}\\
\text{replacing } \frac{\partial u}{\partial t} &\text{ by } \frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t}\\
\text{replacing } \frac{\partial^2 u}{\partial x^2} &\text{ by } \frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{(\Delta x)^2}\\
\text{replacing } \frac{\partial u}{\partial x} &\text{ by } \frac{u_{j+1}^{(m)} - u_{j-1}^{(m)}}{2\Delta x}\\
\text{replacing } u &\text{ by } u_j^{(m)}
\end{aligned}
$$

replacing other functions by their values at (x_j, t_m).

So we can compute $u_j^{(m+1)}$ $(1 \le j \le N-1)$, from previous layers, then we compute $u_0^{(m+1)}$ and $u_N^{(m+1)}$ from boundary conditions by:

$$
\begin{aligned}
\text{replacing } u(0,t) &\text{ by } u_0^{(m+1)}\\
\text{replacing } \frac{\partial u}{\partial x}(0.t) &\text{ by } \frac{u_1^{(m+1)} - u_0^{(m+1)}}{\Delta x}\\
\text{replacing } u(L,t) &\text{ by } u_N^{(m+1)}\\
\text{replacing } \frac{\partial u}{\partial x}(L.t) &\text{ by } \frac{u_N^{(m+1)} - u_{N-1}^{(m+1)}}{\Delta x}
\end{aligned}
$$