D 1. Consider three events E, F and G. Assume that they have the following probabilities: $Pr(E) = \frac{1}{2}$, $Pr(F) = \frac{9}{20}$, and $Pr(G) = \frac{2}{5}$. Assume that we also know the following conditional probabilities $Pr(E|F) = \frac{2}{3}$, $Pr(E|G) = \frac{5}{8}$, $Pr(F|G) = \frac{3}{8}$, and $Pr((E \cap G)|F) = \frac{4}{9}$. Then $Pr(F \cup (E \cap G))$ is (There may be more information than is needed to solve the problem):

A. 1/4
B. 1/10
C. 16/25
D. 1/2
E. 2/9
F. 7/10
G. 3/4
H. 19/20

$$Pr(F \cup (E \cap G)) = Pr(F) + Pr(E \cap G) - Pr(F \cap (E \cap G)) \quad (*)$$

$$Pr(F) = \frac{9}{20}$$

$$Pr(E \cap G) = Pr(E|G) \cdot Pr(G) = \frac{5}{8} \times \frac{2}{5} = \frac{1}{4}$$

$$Pr(F \cap (E \cap G)) = Pr(F \cap E \cap G) = Pr(E \cap G \cap F) = Pr(E \cap G | F)\ Pr(F) = \frac{4}{9} \times \frac{9}{20} = \frac{1}{5}$$

Plug into equation (*), we get $Pr(F \cup (E \cap G)) = \frac{9}{20} + \frac{1}{4} - \frac{1}{5} = \frac{1}{2}$

choose D.

C 2. Four socks are randomly selected from among ten socks, six of which are red and four are black. What is the probability that the four socks will be a pair of red socks and a pair of black socks?

A. 4/6 B. 4/10 C. 3/7 D. 1/90 E. $90/2^{10}$ F. 1/14 G. 4/100 H. 6/10

Size of sample space is $\binom{10}{4}$,

~~In order~~ To get 2 red socks and 2 black socks,

we have $\binom{6}{2} \times \binom{4}{2}$ combinations

So the probability is $\dfrac{\binom{6}{2} \times \binom{4}{2}}{\binom{10}{4}} = \dfrac{3}{7}$

Choose C

D 3. The random variable X denotes the maximum number showing when two fair dice are tossed. (For example if a 2 and a 4 are tossed then $X = 4$.) Which of the following is closest to the expected value of X?

A. 1.5 B. 2.5 C. 3.5 D. 4.5 E. 5.5 F. 6.5 G. 0.5 H. 0

Denote (a, b) as the outcome of the toss (count order)

then we have $X = \max(a, b)$, and we get the following table

(a, b)	X	Pr
(1, 1)	1	$Pr(X=1) = \frac{1}{36}$
(1, 2), (2,2), (2, 1)	2	$Pr(X=2) = \frac{3}{36}$
(1, 3), (2,3) (3,3) (3,1), (3,2)	3	$Pr(X=3) = \frac{5}{36}$
(1,4), (2,4), (3,4), (4,4), (4,1), (4,2) (4,3)	4	$Pr(X=4) = \frac{7}{36}$
(1,5), (2,5), (3,5) (4,5), (5,5), (5,1) (5,2), (5,3), (5,4)	5	$Pr(X=5) = \frac{9}{36}$
(1,6), (2,6), (3,6) (4,6), (5,6), (6,6) (6,1), (6,2), (6,3), (6,4), (6,5)	6	$Pr(X=6) = \frac{11}{36}$

So the expectation is

$$1 \times \frac{1}{36} + 2 \times \frac{3}{36} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36}$$

$$= \frac{161}{36} = 4 + \frac{17}{36} \approx 4.5$$

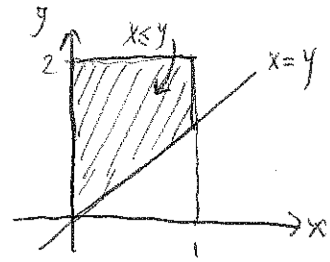
Choose D.

G 4. Let the bivariate distribution of two continuous random variable X and Y be given by the joint probability density function:

$$f(x,y) = \begin{cases} xy & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & otherwise \end{cases}$$

What is $Pr(X \le Y)$?

A. 1/8 B. 1/4 C. 3/8 D. 1/2 E. 5/8 F. 3/4 G. 7/8 H. 1



$Pr(X \le Y)$

$= \iint xy \, dx \, dy$

$= \int_0^1 \int_x^2 xy \, dy \, dx$

$= \int_0^1 \frac{1}{2} x y^2 \Big|_x^2 \, dx$

$= \int_0^1 2x - \frac{1}{2} x^3 \, dx$

$= x^2 - \frac{1}{8} x^4 \Big|_0^1$

$= \frac{7}{8}$

choose G.

The shaded part is described as

$x \le y \le 2$

$0 \le x \le 1$

A

5. You are given 12 balls numbered 1 through 12. You are to place them in two boxes, a red box and a green box, so that each box has 3 even numbered balls and 3 odd numbered balls. How many ways can you do this?

A. 400 B. 200 C. 9 D. 6 E. 18 F. 100 G. 12 H. 2400

We have 6 even numbered balls and 6 odd numbered balls. First we choose 3 even and 3 odd balls for the red box, we have $\binom{6}{3} \times \binom{6}{3}$ ways. Then the rest of balls goes into the green box, and there are 3 even and 3 odd balls, which satisfies the requirement. so there are $\binom{6}{3} \times \binom{6}{3}$ ways,

$= 20 \times 20$

$= 400.$

Choose A.

B 6. Four kids are shooting free throws in the back yard. They take turns shooting such that they are equally likely to be shooting. Alice makes 20% of her shots, Bruce makes 20%, Chris makes 30%, and Daisy makes 50%. You see a ball go through the hoop (i.e. a shot is made). What is the probability that the shooter was Bruce?

A. 1/2
B. 1/6
C. 2/5
D. 1/3
E. 1/4
F. 1/5
G. 3/10
H. 1/10

Use Baye's formula.

$$P(\text{Bruce} \mid \text{shot}) = \frac{P(\text{shot} \mid \text{Bruce})\, P(\text{Bruce})}{P(\text{shot} \mid \text{Alice})P(\text{Alice}) + P(\text{shot} \mid \text{Bruce})P(\text{Bruce}) + P(\text{shot} \mid \text{Chris})P(\text{Chris}) + P(\text{shot} \mid \text{Daisy})P(\text{Daisy})}$$

$$= \frac{20\% \times \frac{1}{4}}{20\% \times \frac{1}{4} + 20\% \times \frac{1}{4} + 30\% \times \frac{1}{4} + 50\% \times \frac{1}{4}}$$

$$= \frac{1}{6}$$

Choose B.

F 7. Men and women enter a store with equal probability. On a given day, what is the probability that the first 4 people entering the store were women if we know that at least 3 (not necessarily the first 3) of the first 4 were women?

A. 1/2
B. 1/3
C. 2/3
D. 1/4
E. 3/4
F. 1/5
G. 2/5
H. 3/5

W : woman

M : man

$$P(\text{wwww} \mid \text{at lease } 3\ w\text{'s}) = \frac{P(\text{wwww})}{P(\text{at least } 3\ w\text{'s})}$$

$$= \frac{P(\text{wwww})}{P(\text{wwww}) + P(\text{wwwM}) + P(\text{wwMw}) + P(\text{wMww}) + P(\text{Mwww})}$$

{at least 3 w's}

= { wwww, wwwM, wwMw, wMww, Mwww }

$$= \frac{(\frac{1}{2})^4}{5 \times (\frac{1}{2})^4} = \frac{1}{5}$$

Choose F.

B 8. Suppose that for some constant C the probability density function, f, of a random variable X is given by: $f(x) = C/\sqrt{1-x}$ for $0 < x < 1$ and 0 otherwise. Find $P(X \leq 1/3)$. (Hint: first find the constant C.)

A. $\sqrt{2/3} - 1$ B. $1 - \sqrt{2/3}$ C. 1 D. $1 - \sqrt{1/3}$
E. 0 F. $\sqrt{1/3}$ G. $\frac{2(\sqrt{3}-\sqrt{2})}{\sqrt{3}}$ H. $1 - \frac{2(\sqrt{3}-\sqrt{2})}{\sqrt{3}}$

We need the density function integrate to 1,

$$\int_0^1 \frac{C}{\sqrt{1-x}}\,dx = 1$$

$$\Rightarrow \quad -2C\sqrt{1-x}\Big|_0^1 = 1$$

$$\Rightarrow \quad 2C = 1$$

$$\Rightarrow \quad C = \frac{1}{2}.$$

So $$P(X \leq \tfrac{1}{3}) = \int_0^{\frac{1}{3}} \frac{\frac{1}{2}}{\sqrt{1-x}}\,dx$$

$$= -\sqrt{1-x}\Big|_0^{\frac{1}{3}}$$

$$= -\sqrt{\tfrac{2}{3}} + 1$$

$$= 1 - \sqrt{\tfrac{2}{3}}$$

Choose B.

C 9. Mary and Bill shoot arrows at a target. The probability of Mary hitting the target on any given shot is 2/5, while the probability of Bill hitting the target on any given shot is 1/4. Mary will shoot first and then they will take turns shooting. Find the probability that the target is hit for the first time on Mary's third try.

A. 1/12 B. 1/250 C. 81/1000 D. 1/10
E. 13/20 F. 9/45 G. 81/2000 H. 81/500

We need the sequence: Mary Miss, Bill Miss, Mary Miss, Bill miss, Mary ~~hit~~ hit ~~hit~~ ~~hit~~

and the corresponding probability is

$$\frac{3}{5} \times \frac{3}{4} \times \frac{3}{5} \times \frac{3}{4} \times \frac{2}{5}$$

$$= \frac{81}{1000}.$$

Choose C

F

10. What is the variance σ^2 of a random variable X distributed on the interval, $[0, 3]$ with (cumulative) distribution function $F(x) = x/3$ for $0 \le x \le 3$.

A. 2/9 B. 1/3 C. 1/2 D. 1/4 E. 4/3 F. 3/4 G. 1 H. 2/3

density function $f(x) = F'(x) = \frac{1}{3}$ $\quad 0 \le x \le 3$

Expectation $E(x) = \int_0^3 x \cdot f(x)\, dx$

$= \int_0^3 \frac{1}{3} x \, dx$

$= \frac{1}{6} x^2 \Big|_0^3$

$= \frac{9}{6}$

$= \frac{3}{2}$

Variance $\sigma^2(x) = \int_0^3 (x - \frac{3}{2})^2 \cdot f(x)\, dx$

$= \int_0^3 (x - \frac{3}{2})^2 \cdot \frac{1}{3}\, dx$

$= \frac{1}{3} \cdot (x - \frac{3}{2})^3 \cdot \frac{1}{3} \Big|_0^3$

$= \frac{1}{9} \times (\frac{3}{2})^3 - \frac{1}{9} \times (-\frac{3}{2})^3$

$= \frac{3}{4}$.

Choose F.